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MOVING LOADS

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2 INFLUENCE LINES

3 INFLUENCE LINES FOR GIRDER WITH FLOOR BEAMS

4 INFLUENCE LINES FOR STRESSES IN FRAMES

5 THE MÜLLER-BRESLAU PRINCIPLE
1.1. INTRODUCTION

In the case of static or fixed load positions, the B.M. and S.F. diagrams can be plotted for a girder, by the simple principles of statics. In the case of rolling loads, however, the B.M. and S.F. at a section of the girder change as the loads move from one position to the other. The problem is, therefore, two-fold:

(i) to determine the load positions for maximum bending moment or shear force for a given section of a girder and to compute its value, and (ii) to determine the load positions so as to cause absolute maximum bending moment or shear force anywhere on the girder.

For every cross-section of girder, the maximum B.M. and S.F. can be worked out by placing the loads in appropriate positions. When these are plotted for all the sections of the girder, we get the maximum B.M. and maximum S.F. diagrams. The ordinate of a maximum B.M. or S.F. diagram at a section gives the maximum B.M. (or S.F.) at that section, due to a given train of loads.

We shall consider the following cases of loadings:
1. Single concentrated load.
2. Uniformly distributed load longer than the span of the girder.
3. Uniformly distributed load shorter than the span of the girder.
4. Two loads with a specified distance between them.
5. Multiple concentrated loads (train of wheel loads).

Sign Conventions

The following sign conventions will be followed for B.M. and S.F. at a given section (Fig. 1.1).

(1) A shear force having an upward direction to the left hand side of a section or downwards to the right of the section will be taken positive. Similarly, a negative S.F. will be one that has a downward direction to the left of the section or

![Diagram of shear force and bending moment with positive and negative signs](image-url)

FIG. 1.1.
upward direction to the right of the section [Fig. 1.1(a, b)].

(2) A B.M. causing concavity upwards will be taken as positive and will be called sagging
B.M. Similarly a B.M. causing convexity upwards will be taken as negative, and will be called
hoggling bending moment [Fig. 1.1(c, d)].

1.2. SINGLE CONCENTRATED LOAD

Let us now consider a single concentrated load \( W \) travelling or rolling along a simply
supported beam or girder \( AB \), of span \( L \), from left to right.

(a) MAXIMUM SHEAR FORCE DIAGRAMS

Consider a point \( C \), distant \( x \) from the left support \( A \). Let the distance of load \( W \) be
\( y \) from \( A \). For any load position, the reaction \( R_B = \frac{Wy}{L} \) and \( R_A = \frac{W(L - y)}{L} \).

(ai) Load in \( AC \) (\( y < x \))

Let the load \( W \) be in \( AC \), such that \( y \) is lesser than \( x \).

Then

\[
F_x \text{ (at C)} = - R_B = - \frac{Wy}{L}
\]

Thus, the shear force \( F_x \) increases as \( y \) increases till \( y = x \), in which case,

\[
F_{\max} = - \frac{Wx}{L} \quad \text{...(1.1)}
\]

This happens when the load is on the section \( C \) itself, thus giving the maximum value of negative shear force
at the section.

For different values of \( x \) (i.e. for different position of the section \( C \)), the
maximum negative shear force given by equation 1.1 will vary linearly with \( x \).

Thus, at \( x = 0 \), \( F_{\max} \) (−) = 0

at \( x = L \),

\[
F_{\max} \text{ (−)} = - \frac{WL}{L} = - W = F_{\max,\max}
\]

The absolute maximum negative S.F., therefore occurs at the right hand support, its value being equal to − \( W \).

The maximum – ve shear force diagram is represented by \( ab_1 \) of Fig. 1.2(b).

(aii) Load in \( CB \) (\( y > x \))

We have seen that when the wheel load \( W \) reaches the section \( C \), maximum –ve S.F.
of value \( \frac{Wx}{L} \) occurs at the section. When the load moves further (i.e. when \( y \) becomes greater
than \( x \)), we have

\[
F_x \text{ (at C)} = + R_A = + \frac{W(L - y)}{L}
\]

Thus, the shear force changes sign immediately when the load crosses the section. The
maximum positive S.F. occurs evidently when \( y = x \) (i.e. when \( y \) is least and the load is in \( CB \)).
Thus, 

\[ F_{\text{max}} = + \frac{W(L - x)}{L} \]  

...(1.2)

For different values of \( x \) (i.e. for different positions of the section \( C \), the maximum positive shear force, given by equation 1.2 will vary linearly with \( x \).

Thus, at \( x = 0 \), \( F_{\text{max}}(+) = + \frac{WL}{L} = W = F_{\text{max max}} \)

at \( x = L \), \( F_{\text{max}}(+) = + \frac{W(L - L)}{L} = 0 \).

The absolute maximum +ve. S.F., therefore, occurs at the left hand support, its value being +W.

The maximum positive S.F.D. is represented by \( a_{1}b \) of Fig. 1.2 (b).

(b) MAXIMUM BENDING MOMENT DIAGRAM

Let us now draw the maximum bending moment diagram for the beam \( AB \). It must be noted that a simply supported beam, under downward loads, bends causing concavity to the upper side. Hence the bending moment is always positive for all sections of the beam. Therefore, the maximum bending moment diagram will also be positive.

(bii) Load in \( AC \) \((y < x)\)

When the load is in \( AC \), \( M_{N} = + R_{A}(L - x) = + \frac{W}{L}y(L - x) \)  

...(1)

This increases as \( y \) increases. When the load reaches the section \( C \), \( y = x \), and the section has the maximum bending moment.

\[ M_{\text{max}} = + \frac{W}{L}x(L - x) \]  

...(1.3)

(bii) Load in \( CB \) \((y > x)\)

When the load \( W \) crosses the section \( C \), \( M_{N} = + R_{A}x = + \frac{W(L - y)}{L}x \)  

...(2)

This increases as \( y \) decreases. When the load is on the section \( C \), \( y = x \), and the section has the maximum bending moment.

\[ M_{\text{max}} = + \frac{W(L - x)x}{L} \]  

which is the same as equation 1.3.

Thus, the maximum bending moment at a section occurs when the load is on the section itself.

For different values of \( x \) (i.e. for different positions of section \( C \)) the maximum bending moment given by equation 1.3 will vary parabolically with \( x \). Fig. 1.2(c) shows the maximum bending moment diagram. For absolute maximum bending moment

\[ \frac{dM_{\text{max}}}{dx} = 0 \]

\[ + \frac{W}{L}(L - 2x) = 0 \quad \text{or} \quad x = \frac{L}{2} \]

Thus, the absolute maximum bending moment occurs at the centre of the span, and its value is given by

\[ M_{\text{max max}} = + \frac{W}{L} \left( L - \frac{L}{2} \right) \frac{L}{2} = + \frac{WL}{4}. \]
1.3. UNIFORMLY DISTRIBUTED LOAD LONGER THAN THE SPAN OF THE GIRDER

Let us now study the case of the uniformly distributed load \( w \) per unit length, longer than the span, and moving from left to right.

(a) MAXIMUM S.F. DIAGRAM

Let us consider a section \( C \) distant \( x \) from left support \( A \), as shown in Fig. 1.3(a). Let the head of the load be distant \( y \) from \( A \).

\[
R_B = \frac{wy^2}{2L}.
\]

When the load is in \( AC \) (i.e. \( y < x \)),

\[
F_x = -R_B = -\frac{wy^2}{2L} \quad \text{...(1)}
\]

This evidently increases as \( y \) increases, until the head of the load reaches the section \( C \) (i.e., when \( y = x \)).

\[
F_{\text{max}} = -\frac{wx^2}{2L} \quad \text{...(1.4)}
\]

When the load still moves further, it can be proved that the value of \( F_x \) given by equation 1.4 decreases. To prove this statement, let the head of the load move by a distance \( \delta x \) from \( C \) towards \( B \), and let \( \delta R_B \) be the corresponding increase in the reaction at \( B \).

Then

\[
R_B + \delta R_B = \frac{w(x + \delta x)^2}{2L}
\]

and

\[
F_x = - (R_B + \delta R_B) + w \cdot \delta x = -\frac{w(x + \delta x)^2}{2L} + w \cdot \delta x
\]

\[
=-\frac{wx^2}{2L} - \left( \frac{wx}{L} \cdot \delta x \right) \quad \text{...(2)}
\]

(neglecting the small quantities of second order).

Since \( \frac{x}{L} < 1 \), the expression inside the bracket is negative. Hence \( F_x \) given by (2) is less than \( F_x \) given by equation 1.4. Thus, the maximum negative shear at a section occurs when the head of the load reaches the section, (i.e. when the left-portion \( AC \) is loaded and the right portion \( CB \) is empty).

The maximum negative S.F. diagram can be plotted by giving different values of \( x \) in equation 1.4.

Thus, at \( x = 0 \), \( F_{\text{max}} = 0 \)

At \( x = L \), \( F_{\text{max}} = -\frac{wL^2}{2L} \).

The absolute maximum negative S.F. occurs at the right hand support. The maximum -ve S.F.D. is shown by \( abb_1 \) in Fig. 1.3(b).
ROLLING LOADS

The S.F. at section C will continue to decrease as the load advances further. When the load covers the entire span,

\[ F_X = + R_A - wX = + \frac{wL}{2} - wX \]  

...(3)

This is positive for the section C to be in the left half of the portion.

Let the load still move on so that the portion CB is fully loaded and portion AC is partially loaded, and we have

\[ F_X = - R_B + w (L - x) \]  

...(4)

In the above expression, the quantity \( w (L - x) \) remains constant as the load still moves further, while \( R_B \) diminishes. Thus, with the onward movement of the load, the positive value of \( F_X \) increases. When the tail of the load reaches the section C, we have

\[ F_X = + R_A = + \frac{w (L - x)^2}{2L} \]

This is the maximum value of positive shear force at C. As the load moves further, \( R_A \) decreases, and hence \( F_X \) decreases.

Thus,

\[ F_{\text{max}} = + \frac{w (L - x)^2}{2L} \]  

...(1.5)

The maximum positive shear force thus occurs when AC is empty and CB is fully loaded. To plot the maximum positive S.F. diagram vary \( x \) from 0 to \( L \).

At \( x = 0 \),

\[ F_{\text{max}} = + \frac{wL^2}{2L} = + \frac{wL}{2} = F_{\text{max max}} \ (\text{+ve}) \]

At \( x = L \),

\[ F_{\text{max}} = 0 \]

The maximum +ve S.F.D. is shown by \( aa_b \) in Fig. 1.3 (b).

(b) MAXIMUM B.M. DIAGRAM

Let the head of the load be in AC, such that \( y < x \). \( \therefore R_B = \frac{wy^2}{2L} \)

\[ M_X = + R_B (L - x) = + \frac{wy^2}{2L} (L - x) \]  

...(1)

The value of \( M_X \) goes on increasing as \( y \) increases till the head of the load reaches the section C, and

\[ M_X = + \frac{wx^2}{2L} (L - x) \]  

...(2)

Let the load now advance further by a small amount \( \delta x \), and let \( \delta B_B \) be the corresponding increase in the reaction at \( B \), such that

\[ R + \delta R_B = \frac{w}{2L} (x + \delta x)^2 \]

Hence

\[ M_X = + (R_B + \delta R_B) (L - x) - w \cdot \delta x \cdot \frac{\delta x}{2} = + \frac{w}{2L} (x + \delta x)^2 (L - x) - \frac{w}{2L} (x + \delta x)^2 \]  

...(3)

This is evidently more than that given by (2). Hence the B.M. at the section C continues to increase as the load moves further, till it occupies the whole span. In that case,

\[ M_X = + \frac{wL}{2} - \frac{wx^2}{2} = + \frac{wx (L - x)}{2} \]

...(4)
As the load still moves further, so that portion \( AC \) is partially loaded, and portion \( CB \) is fully loaded, we have

\[
M_X = + R_B (L - x) - \frac{w (L - x)^2}{2} \quad \text{(5)}
\]

In the above expression, the quantity \( \frac{w (L - x)^2}{2} \) is constant, while \( R_B \) diminishes as the load moves further. Hence \( M_X \) decreases till the tail of the load reaches the section \( C \). When the load is in the portion \( CB \) only, \( AC \) is empty.

\[
M_X = + R_A x \quad \text{(6)}
\]

Since \( R_A \) decreases as the load moves further, \( M_X \) also decreases. It can thus be concluded that the values of \( M_X \) given by (5) and (6) are less than that given by (4). Thus, the maximum B.M. at the section occurs when the whole span is loaded, and its value is given by

\[
M_{max} = + \frac{wx (L - x)}{2} \quad \text{(1.5)}
\]

The maximum bending moment diagram is evidently a parabola as shown in Fig. 1.3 (c). The absolute maximum bending moment evidently occurs at the centre of the span \( (x = L/2) \).

Thus,

\[
M_{max, max} = + \frac{wL}{2} \left( L - \frac{L}{2} \right) = + \frac{wL^2}{8}
\]

1.4. UNIFORMLY DISTRIBUTED LOAD SHORTER THAN THE SPAN OF THE GIRDER

Let the uniformly distributed load \( w \) unit length extend over a length \( a \) such that \( a < L \).

(a) MAXIMUM S.F. DIAGRAMS

(ii) Maximum Negative S.F.

(1) \textit{Let the position of the section C be such that } x < a.

When the head of load reaches the section \( C \), the portion \( AC \) is fully loaded (since \( a > x \)).

\[
F_{max} = - R_B = - \frac{wx^2}{2L} \quad \text{(1.6 a)}
\]

This is a parabolic relation and is valid for values of \( x \) between 0 and \( a \). At \( x = 0 \), \( F_{max} = 0 \).

and at \( x = a \), \( F_{max} = - \frac{wa^2}{2L} \)

(2) \textit{Let the position of the section C be such that } x > a.

When the load is in \( AC \), the portion \( AC \) is partially loaded, and \( F_x = - R_B \), which goes on increasing as the head of the load approaches \( C \). When the head of the load reaches \( C \), we have

\[
F_{max} = - R_B = - \frac{wa}{L} \left( x - \frac{a}{2} \right) \quad \text{(1.6} \)
\]

This is a straight line relation.
At \( x = \frac{a}{2} \), \( F_{\text{max}} = 0 \)

At \( x = a \), \( F_{\text{max}} = -\frac{wa}{L} \left( a - \frac{a}{2} \right) = -\frac{wa^2}{2L} \)

At \( x = L \), \( F_{\text{max}} = -\frac{wa}{L} \left( L - \frac{a}{2} \right) \)

The maximum \(-\)ve S.F.D. thus consists of a parabola up to a distance of \( a \) from \( A \), and then straight line up to \( B \).

(aii) Maximum Positive S.F.

As the load moves further, the S.F. decreases. For a particular load position, it becomes zero, and then changes sign and becomes positive. As the load still moves further, the positive S.F. at \( C \) increases. For maximum positive shear force at \( C \), the span \( AC \) should be empty and the reaction at \( A \) should be a maximum. In other words, the tail of the load should be at \( C \), and the load should extend from \( C \) towards \( B \).

When the tail of the load is at \( C \),
\[
F_{\text{max}} = +R_A = +\frac{wa}{L} \left( L - x - \frac{a}{2} \right) \quad \text{...(1.7)}
\]

This is the equation of a straight line, and is valid for all values of \( x \) between 0 to \( (L - a) \).

Thus, at \( x = 0 \), \( F_{\text{max}} = +\frac{wa}{L} \left( L - \frac{a}{2} \right) \)

At \( x = (L-a) \), \( F_{\text{max}} = +\frac{wa}{L} \left( L - L + a - \frac{a}{2} \right) = +\frac{wa^2}{2L} \).

When the position of the section \( C \) is such that \( x > L-a \)

[i.e. when \( x \) varies from \( (L-a) \) to \( L \)]
\[
F_{\text{max}} = +R_A = +\frac{w}{2L} (L - x)^2 \quad \text{...(1.8)}
\]

Thus, \( F_{\text{max}} \) is independent of \( a \), and varies parabolically.

At \( x = L-a \), \( F_{\text{max}} = +\frac{w}{2L} (L - L + a)^2 = +\frac{wa^2}{2L} \), as before

At \( x = L \), \( F_{\text{max}} = +\frac{w}{2L} (L - L)^2 = 0 \).

Thus, the maximum positive shear force diagram is a straight line from \( x = 0 \) to \( x = L-a \), and a parabola between \( x = L-a \) to \( x = L \).

The absolute maximum negative S.F. occurs at support \( B \), when the head of the load is at \( B \), and the absolute maximum positive S.F. is at \( A \) when the tail of the load is at \( A \).

The maximum negative and positive S.F. diagrams have been shown in Fig. 1.4(b).

(b) MAXIMUM B.M. DIAGRAM

Let the length of the U.D.L. be \( a \). When the load is in the portion \( AC \), the B.M. at the section \( C \) is given by
\[
M_x = +R_B (L - x)
\]

This goes on increasing as the head of the load approaches the section \( C \). When the head of the load crosses the section \( C \), the B.M. still goes on increasing, till it attains maximum value at a specific load position. On further movement, the B.M. at the section \( C \) decreases.
For the maximum B.M. at the section C the load is to be so arranged that its C.G. is at a distance \( y \) from \( A \), as shown in Fig. 1.5(a).

In this load position, \( R_B = wa \frac{y}{L} \)

Distance \( CB_1 = (y - x + \frac{a}{2}) \)

\[
M_x = + R_B (L - x) - \frac{wa}{L} (CB_1)^2
\]

\[
= + \frac{wa}{L} (L - x) - \frac{w}{2} (y - x + \frac{a}{2})
\]

... (2)

For \( M_x \) to be maximum, differentiate it with respect to \( y \) and equate to zero. Thus, we have

\[
\frac{dM_x}{dy} = 0 = + \frac{wa}{L} (L - x) - w (y - x + \frac{a}{2})
\]

or

\[
\frac{a}{L} (L - x) = (y - x + \frac{a}{2})
\]

... (1.9)

In the above equation, \( a = A_1B_1 \); \( L = AB \); \( L - x = CB \)

and

\[
y - x + \frac{a}{2} = CB_1
\]

Hence equation 1.9 can be expressed geometrically as

\[
\frac{A_1B_1}{AB} \cdot CB = CB_1
\]

or

\[
\frac{CB}{CB_1} = \frac{AB}{A_1B_1} = \frac{AB - CB}{A_1B_1 - CB_1} = \frac{AC}{A_1C}
\]

or

\[
\frac{A_1C}{CB_1} = \frac{AC}{CB}
\]

... (1.10)

Thus, for maximum bending moment at a section, the load position is such that the section divides the load in the same ratio as it divides the span. This relation will be found to hold good generally, both for the point loads as well as the uniformly distributed loads.

Equation 1.9 is directly useful for the location of the U.D.L. for the maximum B.M.

For the maximum B.M. at \( C \), we get, from equation 1.9,

\[
y = \frac{a}{L} (L - x) + x - \frac{a}{2} = \frac{a}{2} + x - \frac{ax}{L}
\]

and

\[
y - x + \frac{a}{2} = \frac{a}{L} (L - x)
\]

Substituting the values in (2), we get

\[
M_{max} = + \frac{wa}{L} (L - x) \left( \frac{a}{2} + x - \frac{ax}{L} \right) - \frac{w}{2} \left( \frac{a}{L} (L - x) \right)^2
\]

\[
= + \frac{wa}{L} (L - x) \left( 1 - \frac{a}{2L} \right)
\]

...(1.11)
The maximum B.M. diagram can now be plotted by giving different values to \( x \) in equation 1.11. Absolute maximum B.M. occurs evidently at the centre, when \( x = L/2 \).

Thus, from equation 1.11,

\[
M_{\text{max, max}} = + \frac{wa}{L} \cdot \frac{L}{2} \left( L - \frac{L}{2} \right) \left( 1 - \frac{a}{2L} \right) = + \frac{wa}{4} \left( L - \frac{a}{2} \right)
\]

The above value can also be obtained by considering Fig. 1.5, and applying the deduction of equation 1.10 independently.

Thus, for maximum bending moment at the centre of the span,

\[
\frac{AC}{CB_1} = \frac{AC}{CB} = \frac{L}{2} \Rightarrow \frac{AC}{CB_1} = \frac{L/2}{L/2} = 1
\]

or

\[
\frac{AC}{CB_1} = \frac{L/2}{L/2} = \frac{a}{2}
\]

In this position, 
\[
R_a = R_b = \frac{wa}{2}
\]

and 
\[
M_{\text{max}} \text{ (at centre)} = M_{\text{max, max}} = + \frac{wa}{2} \cdot \frac{L}{2} - \frac{w}{2} \left( \frac{a}{2} \right)^2 = + \frac{wa}{4} \left( L - \frac{a}{2} \right)
\]

which is same as before.

1.5. TWO POINT LOADS WITH A FIXED DISTANCE BETWEEN THEM

Let us now consider two point loads, \( W_1 \) and \( W_2 \) at a fixed distance \( d \) apart, moving from left to right with \( W_1 \) leading. Let the leading load \( W_1 \) be smaller than \( W_2 \).

(a) MAXIMUM NEGATIVE SHEAR FORCE

For negative shear force at the section \( C \), we have to consider the three load positions:

1. Both loads to the left of the section \( C \).
2. Load \( W_1 \) to the right of \( C \) and \( W_2 \) to the left of it.
3. Both loads to the right of \( C \) : For this load position, there will be no negative shear (since \( F_X = + R_a \)), and hence we will consider only the first two load positions.

(ii) Both Loads to the Left of \( C \)

For this load position,

\[
F_X = -R_b = \ldots (i)
\]

This increases as the leading load reaches near the section \( C \), and is maximum when \( W_1 \) is just to the left of \( C \).

1. When \( x < d \), only \( W_1 \) will be on the girder and \( W_2 \) will be off the span, with \( W_1 \) at \( C \). Hence,

\[
0^1 F_{\text{max}} = -R_b = - \frac{W_1 x}{L} \quad \ldots (i) \ldots (1.12)
\]

2. When \( x > d \), both \( W_1 \) and \( W_2 \) will be on the girder with \( W_1 \) at \( C \). Hence

\[
1^1 F_{\text{max}} = -R_b = - \frac{W_1 x + W_2 (x - d)}{L} \quad \ldots (ii) \ldots (1.13)
\]
(a) $W_1$ to the Right of $C$ and $W_2$ to the Left of $C$

For this load position,

\[ F_X = -R_B + W_1 \]

...(2)

Since $R_B$ increases as $W_1$ and $W_2$ reach near $B$, the maximum S.F. occurs when the load $W_2$ is just to the left of $C$.

(3) When $(L - x) > d$, both $W_1$ and $W_2$ will be on the beam with $W_2$ at $C$. Hence

\[ F_{max} = -R_B + W_1 \]

\[ = -\frac{W_2 x + W_1 (x+d)}{L} + W_1 \]

...(III) ...(1.14)

(4) When $(L - x) < d$, load $W_1$ will be off the girder while $W_2$ is at $C$. Hence

\[ F_{max} = -R_B = -\frac{W_2 x}{L} \]

...(IV) ...(1.15)

Thus, we have four equations for $F_X$ (equations I, II, III and IV) and one or the other of these equations will give maximum negative shear force depending upon the relative magnitudes of $x$ and $d$.

To find which of these four equations will give $F_{max}$ we shall divide the beam in three zones:

(i) Zone (1):
\[ x = 0 \text{ to } x = d \]

(ii) Zone (2):
\[ x = d \text{ to } x = (L - d) \]

(iii) Zone (3):
\[ x = (L - d) \text{ to } x = L \]

(i) Zone (1): $x = 0$ to $x = d$.

The first zone under consideration is from $x = 0$ to $x = d$, and for this, both equations I as well as III will be applicable. For equation (I), $W_1$ is at $C$ while $W_2$ is off the girder. For equation III, $W_2$ is at $C$ and $W_1$ is to the right of it. Out of the two, equation I will give the larger value if

\[ \frac{W_1 x}{L} > \frac{W_2 x + W_1 (x+d)}{L} - W_1 \]
or
\[ x < \frac{W_1(L - d)}{W_2} \]  

...(1.16)

Thus, when \( x < \frac{W_1(L - d)}{W_2} \), equation I will give greater \( F_{\text{max}} \).

Beyond this value \( \{ \text{i.e., } x = \frac{W_1(L - d)}{W_2} \text{ to } x = d \} \) equation III will give greater \( F_{\text{max}} \).

(ii) Zone (2) \( \{ x = d \text{ to } x = L - d \} \)

The second zone under consideration is from \( x = d \) to \( x = L - d \), and for this both equations II as well as III will be applicable. For Eq. (II), \( W_1 \) is at \( C \) and \( W_2 \) is to the left of it. For Eq. (III), \( W_2 \) is at \( C \) and \( W_1 \) is to the right of it. Out of the two, equation (II) will give larger value if

\[ \frac{1}{F_{\text{max}}} > \frac{1}{F_{\text{max}}} \]

i.e.
\[ \frac{W_1x + W_2(x - d)}{L} > \frac{W_1x + W_2(x + d) - W_1}{L} \]

or
\[ d < \frac{W_1L}{W_1 + W_2} \]

...(1.17)

Thus when the value of \( d \) is less than \( \frac{W_1L}{W_1 + W_2} \), max - ve S.F. will occur when the leading load reaches the section. This is the standard case for which \( \text{max. S.F.D. has been drawn in Fig. 1.6(b).} \)

Thus, when \( d < \frac{W_1L}{W_1 + W_2} \), equation II will give \( F_{\text{max}} \).

When \( d > \frac{W_1L}{W_1 + W_2} \), equation III will give \( F_{\text{max}} \) and maximum - ve S.F. will occur when the rear wheel load reaches the section.

Thus from Eq. I, when \( x = 0 \), \( F_{\text{max}} = 0 \); when \( x = d \), \( F_{\text{max}}(-) = \frac{W_1d}{L} \)

From Eq. II when \( x = d \), \( F_{\text{max}} = \frac{W_1d}{L} \); when \( x = L \), \( F_{\text{max}}(-) = W_1 + \frac{W_2(L - d)}{L} \)

(b) MAXIMUM POSITIVE SHEAR FORCE

In this case also, we will consider the three load positions for maximum positive shear force at the section (C):

(1) Both loads to the right of C.
(2) \( W_1 \) to the right of \( C \) and \( W_2 \) to the left of \( C \).
(3) Both loads to the left of \( C \) : For this position, there will be no positive S.F. (Since \( F_X = - R_B \)) and hence we will consider only the first two load positions.

(i) Both loads to the right of \( C \)

For this load position,

\[ F_X = + R_A \]

...(I)

This increases when \( R_A \) increases. Hence the maximum value occurs when \( W_2 \) is just to the right of \( C \).
(1) When \((L - x) > d\), both \(W_2\) and \(W_1\) will be on the girder, with \(W_2\) at \(C\). Hence
\[
^2F_{max} = + R_A = + \frac{W_2(L - x) + W_1(L - x - d)}{L} \quad ... (V) ... (1.18)
\]

(2) When \((L - x) < d\), only \(W_2\) will be at \(C\) and \(W_1\) will be off the girder. Hence
\[
^0F_{max} = + R_A = + \frac{W_2(L - x)}{L} \quad ... (VI) ... (1.19)
\]

(bii) \(W_1\) to the right of \(C\) and \(W_2\) to the left

In this case, maximum positive S.F. occurs when \(W_1\) is just to the right of \(C\).

(3) When \(x < d\), the load \(W_2\) will be off the girder with load \(W_1\) just to the right of \(C\). Hence
\[
^0F_{max} = + R_A = + \frac{W_1(L - x)}{L} \quad ... (VII) ... (1.20)
\]

(4) When \(x > d\), both \(W_1\) and \(W_2\) will be on the girder with \(W_1\) just to the right of \(C\). Hence
\[
^1F_{max} = + \frac{W_1(L - x) + W_2(L - x + d)}{L} - W_2 \quad ... (VIII) ... (1.21)
\]

Equations V, VI, VII and VIII are valid for appropriate ranges of \(x\).

For the case when \(W_1 < W_2\), equation V will give the maximum positive shear force, and is valid for all values of \(x\) between 0 to \((L - d)\). Beyond this, \(W_1\) is off the girder, and equation VI will be valid.

Thus, at \(x = 0\),
\[
^2F_{max} = + \left( W_2 + \frac{W_1(L - d)}{L} \right)
\]

At \(x = (L - d)\),
\[
^2F_{max} = + \frac{W_2d}{L}
\]

The complete maximum S.F.D. has been shown in Fig. 1.6 (b). Eqs. VII and VIII will give maximum value only when \(W_1 < W_2\), i.e., when the loads move in the reverse order.

Summary:

(1) The maximum negative S.F. occurs only when both the loads are to the left of the section with \(W_1\) just approaching it. This is valid if \(d < \frac{W_1L}{W_1 + W_2}\). The maximum – ve S.F.D. is governed by Eqs. I and II.

(2) If \(d > \frac{W_1L}{W_1 + W_2}\), maximum negative S.F. occurs when \(W_2\) is just to the left of \(C\), and \(W_1\) is to the right of \(C\). The maximum – ve S.F.D. is governed by Eqs. I and III.

(3) The maximum positive S.F. occurs only when both the loads are to the right of the section. The maximum positive S.F.D. is governed by Eqs. V and VI.

(4) If \(W_2\) is greater than \(W_1\) (i.e., when the loads travel in reverse order), Eqs. VII and VIII are valid for maximum positive S.F.D.

See example 1.2 and 1.3 for complete illustration.

(c) MAXIMUM BENDING MOMENT DIAGRAM

When the two loads \(W_1\) and \(W_2\) are to the left of section \(C\),
\[
M_X = + R_B (L - x) \quad ... (1)
\]
This goes on increasing, as \( R_b \) increases, till \( W_1 \) reaches the section \( C \). Let \( ^1M_X \) be the bending moment at \( C \) when \( W_1 \) is on the section and \( W_2 \) is to the left of it.

Then,
\[
^1M_X = + R_b (L - x) = + \left[ \frac{W_1 x + W_2 (x - d)}{L} \right] (L - x) \quad \ldots (I) \ldots (1.22)
\]

When both the loads are to the right of section \( C \),
\[
M_X = + R_A \cdot x
\]

This is evidently maximum when \( W_2 \) is at \( C \) and \( W_1 \) is ahead of it. Let \( ^2M_X \) be the bending moment at \( C \) when \( W_2 \) is on it and \( W_1 \) is to the right of it.

Then,
\[
^2M_X = + R_A \cdot x = + \frac{x}{L} \left( W_1 (L - x - d) + W_2 (L - x) \right) \quad \ldots (II) \ldots (1.23)
\]

As the loads still move further, \( R_A \) decreases, and hence \( ^2M_X \) decreases.

As a third possibility of getting maximum bending moment at \( C \), let \( W_1 \) be to the right of \( C \), and \( W_2 \) to the left of it at a distance \( y \) from \( C \). Let \( ^3M_X \) be the bending moment at \( C \), for this loading. Then,
\[
^3M_X = + R_b (L - x) - W_1 (d - y)
\]
\[
= + \left[ \frac{W_2 (y - x) + W_1 (x - y + d)}{L} \right] (L - x) - W_1 (d - y)
\]

The above equation may be rewritten in terms of \( ^1M_X \) and \( ^2M_X \) as under:
\[
^3M_X = ^1M_X - \frac{d - y}{L} \left[ W_1 x - W_2 (L - x) \right]
\]
and
\[
^3M_X = ^2M_X + \frac{y}{L} \left[ W_1 x - W_2 (L - x) \right]
\]

If \( W_1 x > W_2 (L - x) \), \( ^1M_X > ^2M_X > ^3M_X \)

If \( W_1 x < W_2 (L - x) \), \( ^3M_X > ^2M_X > ^1M_X \)

It is clear, therefore, that in either case, \( ^3M_X \) will not be maximum.

Hence maximum B.M. at the section is either \( ^1M_X \) or \( ^2M_X \), whichever is larger. Fig. 1.6 (c) shows both the parabolas giving \( ^1M_X \) and \( ^2M_X \) governed by equations I and II respectively.

Now \( ^1M_X \) is greater than \( ^2M_X \) if
\[
\frac{W_1 x + W_2 (x - d)}{L} (L - x) > \frac{W_1 (L - x - d) + W_2 (L - x)}{L}
\]

i.e.
\[
x > \frac{W_1 L}{W_1 + W_2} \quad \ldots (1.24)
\]

For \( x < \frac{W_2 L}{W_1 + W_2} \), \( ^2M_X \) is maximum.

For \( x > \frac{W_1 L}{W_1 + W_2} \), \( ^1M_X \) is maximum.

Now, \( ^1M_X \) is zero at \( x = \frac{W_2 d}{W_1 + W_2} \), and at \( x = L \).

\( ^2M_X \) is zero at \( x = 0 \), and at \( (L - x) = \frac{W_1 d}{W_1 + W_2} \).
Both the parabolas cross each other at \( F' \), where \( ^1M_x = ^2M_x \). To find the position of this section, put \( ^1M_x = ^2M_x \).

Thus

\[
\frac{W_1x + W_2(x - d)}{L} (L - x) = \frac{W_1}{L} (L - x - d) + \frac{W_2}{L} (L - x) x
\]

From which \( x \) (or \( AF \)) = \( \frac{W_2 L}{W_1 + W_2} \).

Thus, \( F \) divides \( AB \) in the ratio of \( W_1 : W_2 \). For all sections from \( A \) to \( F \), maximum B.M. is given by \( ^1M_x \), and for all sections from \( F \) to \( B \), the maximum B.M. is given by \( ^2M_x \).

The maximum value of \( ^2M_x \) in equation II will occur at

\[
x = \frac{AE}{2} = \frac{1}{2} \left( L - \frac{W_1d}{W_1 + W_2} \right)
\]

When \( W_2 > W_1 \), then absolutely maximum B.M. anywhere in the girder occurs in the \( ^2M_x \) range at \( x = \frac{1}{2} AE \).

In case \( (L - x) < d \), \( W_1 \) is off the girder when \( W_2 \) is on the section. Let \( ^0M_x \) be the bending moment at \( C \) when \( W_2 \) is on it and \( W_1 \) is off the girder.

Then

\[
^0M_x = + R_b (L - x) = + \frac{W_2}{L} x (L - x) \quad \text{... (III) (1.25)}
\]

Similarly, when \( x < d \), \( W_2 \) is off the girder when \( W_1 \) is on the section. Let \( ^1M_x \) be the bending moment at \( C \) when \( W_1 \) is on it and \( W_2 \) is off the girder.

Then

\[
^1M_x = + R_b (L - x) = + \frac{W_1}{L} x (L - x) \quad \text{... (IV) (1.26)}
\]

For some sections over a portion of the girder, \( ^0M_X \) may sometimes be greater than \( ^1M_x \). Fig 1.6(d) and (e) show the B.M.D. for two such possibilities. See also examples 1.3 and 1.4 for such possibilities.

**Example 1.1.** A uniformly distributed load of 1 kN per meter run, 6 m long crosses a girder of 16 m span. Construct the maximum S.F. and B.M. diagrams and calculate the values at sections 3 m, 5 m and 8 m from the left hand support.

**Solution.**

(a) Maximum Negative S.F. Diagram.

The maximum negative and positive S.F. diagrams are plotted exactly in the same manner as explained in § 1.4.

For \( x \) upto 6 m (=a)

\[
F_{max} = - \frac{wx^2}{2L} = - \frac{1 \times x^2}{2 \times 16} = - \frac{x^2}{32} \text{ kN} \quad \text{... (1.6 a)}
\]

At

\[
x = 0, \quad F_0 = 0
\]

\[
x = 3 \text{ m}, \quad F_3 = - \frac{9}{32} \text{ kN}
\]

\[
x = 5 \text{ m}, \quad F_5 = - \frac{25}{32} \text{ kN}
\]

\[
x = 6 \text{ m}(=a), \quad F_6 = - \frac{36}{32} = - \frac{9}{8} \text{ kN}
\]
For $x > 6$ m

$$F_{max} = \frac{wa}{L} \left( x - \frac{a}{2} \right)
= -\frac{1 \times 6}{16} \left( x - \frac{6}{2} \right) = -\frac{3}{8} (x-3)
$$

Linear variation \hspace{1cm} ...(1.6)

At $x = 6$, $F_0 = -\frac{3}{8} (6 - 3) = -\frac{9}{8}$, as before.

$x = 8$, $F_8 = -\frac{3}{8} (8 - 3) = -\frac{15}{8}$ kN

$x = 16$, $F_{16} = F_{max} = -\frac{3}{8} (16-3)
= -\frac{39}{8}$ kN

(b) Maximum Positive S.F.

Diagram

(1) For $x$ between 0 to ($L - a$) = 16 - 6 = 10 m

$$F_{max} = + R_A = + \frac{wa}{L} \left( L - x - \frac{a}{2} \right)
= +\frac{1 \times 6}{16} \left( 16 - x - \frac{6}{2} \right) = +\frac{3}{8} (13 - x)
$$

Linear variation \hspace{1cm} ...(1.7)

At $x = 0$, $F_0 = +\frac{3}{8} (13 - 0) = +\frac{39}{8}$ kN

$x = 3$ m, $F_3 = +\frac{3}{8} (13 - 3) = +\frac{15}{4}$ kN

$x = 5$ m, $F_5 = +\frac{3}{8} (13 - 5) = +3$ kN

$x = 8$ m, $F_8 = +\frac{3}{8} (13 - 8) = +\frac{15}{8}$ kN

$x = 10$ m, $F_{10} = +\frac{3}{8} (13 - 10) = +\frac{9}{8}$ kN

(2) For $x$ between 10 m to 16 m

$$F_{max} = + R_A = + \frac{w}{2L} (L-x)^2
= + \frac{1}{2} \times \frac{1}{16} (16 - x)^2 = + \frac{1}{32} (16 - x)^2
$$

Parabolic \hspace{1cm} ...(1.8)

At $x = 10$ m, $F_{10} = + \frac{1}{32} (16 - 10)^2 = + \frac{36}{32} = +\frac{9}{8}$, as before

$x = 16$ m, $F_{16} = + \frac{1}{32} (16 - 16)^2 = 0$.

(c) Maximum Bending Moment

For getting maximum bending moment at a section, the load should be so arranged that the section divides it in the same ratio as it divides the span. Thus, from equation 1.10,

$$\frac{A_1 C}{C B_1} = \frac{AC}{CB}$$

Here \hspace{1cm} $AC = x$ ; \hspace{1cm} $CB = L - x$. 
\[ A_1C = \frac{x}{L-x} \]

or
\[ \frac{A_1C + CB_1}{CB_1} = \frac{A_1B_1}{CB_1} = \frac{x + L - x}{L} = \frac{L}{L-x} \]

or
\[ CB_1 = \frac{L-x}{L} \times A_1B_1 = \frac{a(L-x)}{L} \]

\[(1)\text{ For } x = 3 \text{ m }\]

\[ CB_1 = \frac{a}{L} (L-x) = \frac{6}{16} (16-3) = \frac{39}{8} \text{ m} \]

\[ A_1C = 6 - \frac{39}{8} = \frac{9}{8} \text{ m} \]

\[ BB_1 = 16 - 3 - \frac{39}{8} = \frac{65}{8} \text{ m} \]

\[ R_A = (6 \times 1) \left( \frac{65}{8} + 3 \right) \frac{1}{16} = \frac{267}{64} \text{ kN} \]

and \[ M_3 = + \frac{267}{64} \times 3 - \frac{1}{2} \times \left( \frac{9}{8} \right)^2 = + 11.9 \text{ kN-m} \]

Alternatively, from equation 1.11, we have
\[ M_{\text{max}} = + \frac{w_{\text{max}}}{L} (L-x) \left( 1 - \frac{a}{2L} \right) \]

\[ M_3 = + \frac{1 \times 6 \times 3}{16} (16-3) \left( 1 - \frac{6}{2 \times 16} \right) = + 11.9 \text{ kN-m} \]

which is the same as before.

\[(2)\text{ For } x = 5 \text{ and } x = 8 \]

For these sections also, equation 1.10 can be used to first locate the position of the U.D.L. and the maximum B.M. can then be calculated, or else equation 1.11 can be used to compute maximum B.M. directly. Thus, from equation 1.11.

\[ M_5 = + \frac{1 \times 6 \times 5}{16} (16-5) \left( 1 - \frac{6}{2 \times 16} \right) = + 16.7 \text{ kN-m} \]

and \[ M_8 = + \frac{1 \times 6 \times 8}{16} (16-8) \left( 1 - \frac{6}{2 \times 16} \right) = + 19.5 \text{ kN-m} = M_{\text{max},\text{max}} \]

The results can now be summarised as below:

<table>
<thead>
<tr>
<th>Section x</th>
<th>Max. -ve S.F.</th>
<th>Max. +ve S.F.</th>
<th>Max. B.M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 m</td>
<td>- \frac{9}{32} kN</td>
<td>+ \frac{15}{4} kN</td>
<td>+ 11.9 kN-m</td>
</tr>
<tr>
<td>5 m</td>
<td>- \frac{25}{32} kN</td>
<td>+ 3 kN</td>
<td>+ 16.7 kN-m</td>
</tr>
<tr>
<td>8 m</td>
<td>- \frac{15}{8} kN</td>
<td>+ \frac{15}{8} kN</td>
<td>+ 19.5 kN-m</td>
</tr>
</tbody>
</table>

Example 1.2. Two point loads of 4 kN and 6 kN spaced 6 m apart cross a girder of 16 m span, the 4 kN load leading from left to right. Construct the maximum S.F. and B.M. diagrams, stating the absolute maximum values.

Solution. Given : \( W_1 = 4 \text{ kN} \); \( W_2 = 6 \text{ kN} \); \( d = 6 \text{ m} \); \( L = 16 \text{ m} \)
\[
\frac{W_1L}{W_1 + W_2} = \frac{4 \times 16}{4 + 6} = 6.4
\]

\[
d = 6 < 6.4
\]

Thus the data is for the standard case, and the S.F.D. will be as that shown in Fig. 1.6 (a).

(a) Maximum – ve S.F. Diagram.

Since \(d < \frac{W_1L}{W_1 + W_2}\), the maximum –ve S.F. at any section will occur under the leading load of 4 kN.

For \(x\) up to 6 m (=d),

\[
F_{\text{max}} = - \frac{W_1x}{L} = -\frac{4x}{16} = -\frac{x}{4} \text{ kN.}
\]

\[
F_6 = -\frac{6}{4} = -1.5 \text{ kN}
\]

For \(x\) more than 6 m (=d), \(^1F_{\text{max}}\) will be greater than \(^2F_{\text{max}}\) and is given by equation 1.13:

\[
^1F_{\text{max}} = -R_0 = -\frac{W_1x + W_2(x - d)}{L} = -\frac{(W_1 + W_2)x - W_2d}{L} = -\frac{(4 + 6)x - 6 \times 6}{16} = -\frac{5x - 18}{8}
\]

At \(x = L = 16\) m, \(F_{\text{max,max}} = -\frac{(5 \times 16) - 18}{8} = -\frac{31}{4} \text{ kN.}\)

(b) Maximum + ve S.F. Diagram

For all sections between \(x = 0\) and \(x = L - d = 16 - 6 = 10\) m, the maximum positive S.F. will be when the load \(W_2\) (=6 kN) is at the section, with \(W_1\) (=4 kN) ahead of it. The variation is given by equation 1.18., i.e.,

\[
^2F_{\text{max}} = + R_A = + \frac{W_2(L - x) + W_1(L - x - d)}{L} \quad \ldots(1.18)
\]

At \(x = 0\),

\[
F_A = F_{\text{max,max}} = + \frac{6(16) + 4(16 - 0 - 6)}{16} = + 8.5 \text{ kN}
\]

At \(x = 10\) m, \(F_{10} = + \frac{6 \times 6 + 4(0)}{16} = + \frac{9}{4} \text{ kN}\)

For all section between \(x = 10\) m to \(x = 16\) m, \(W_1\) will be off the girder when \(W_2\) is on it. Hence, max. +ve S.F. is given by equation 1.19 i.e.

\[
^2F_{\text{max}} = + R_A = + \frac{W_2(L - x)}{L} \quad \ldots(1.19)
\]

At \(x = 10\) m,

\[
F_{10} = + \frac{6(16 - 10)}{16} = + \frac{9}{4} \text{ as before,}
\]

At \(x = 16\) m, \(F_B = 0\).

The max. –ve and +ve S.F.D. have been shown in Fig. 1.8(b).

(c) Maximum Bending Moment Diagram

As discussed earlier, the maximum B.M. at a section may occur under any one of the following conditions:
(i) Max. B.M. under $W_1$ and $W_2$ behind it ($^1M_X$).

(ii) Max. B.M. under $W_2$ with $W_1$ ahead of it ($^2M_X$).

(iii) Max. B.M. under $W_2$ with $W_1$ off the girder ($^3M_X$).

We shall investigate all the three possibilities.

(a) Max. B.M. under $W_1$ ($^1M_X$):

From equation 1.22, we have

$$^1M_X = \frac{W_1 x + W_2 (x - d)}{L} (L - x)$$

$$= + \frac{4x + 6(x - 6)}{16} (16 - x)$$

$$= + (10x - 36) \left(1 - \frac{x}{16}\right)$$

This is zero at $x = 3.6$ m and $x = 16$ m.

Thus, $DB = 16 - 3.6 = 12.4$ m.

$\therefore ~ ^1M_{\text{max}}$ will occur at $x = 3.6 + \frac{12.4}{2} = 9.8$ m, its value being

$$^1M_{\text{max}} = (98 - 36) \left(1 - \frac{9.8}{16}\right) = 24.25 \text{ kN-m}.$$ 

(b) Max. B.M. under $W_2$ ($^2M_X$):

From equation 1.23, we have

$$^2M_X = + \frac{x}{L} \left[ W_1 (L - x - d) + W_2 (L - x) \right]$$

$$= + \frac{x}{16} \left[ 4(16 - x - 6) + 6(16 - x) \right] = + \frac{x}{16} \left[ 136 - 10x \right]$$

This is zero at $x = 0$, and $x = 13.6$ m. Thus $AE = 13.6$ m

$^2M_{\text{max}}$ occurs at $x = \frac{AE}{2} = 6.8$ m, its value being

$$^2M_{\text{max}} = + \frac{6.8}{16} \left(136 - 68\right) = + 28.8 \text{ kN-m}.$$ 

Since $\frac{W_2 L}{W_1 + W_2} = \frac{6 \times 16}{6 + 4} = 9.6$ m, for $x < 9.6$ m, $^2M_X$ is greater than $^1M_X$, and for $x > 9.6$ m, $^1M_X$ is greater than $^2M_X$.

At $x = 9.6$ m,

$$^2M_X = + \frac{9.6}{16} \left(136 - 96\right) = + 24 \text{ kN-m}$$

$$^1M_X = (96 - 36) \left(1 - \frac{9.6}{16}\right) = + 24 \text{ kN-m}$$

Thus $^2M_X$ and $^1M_X$ are equal at $x = 9.6$ m (This is a check).
(c) Max. B.M. under \( W_2 \) when \( W_1 \) is off the girder \( (\delta^2 M_x) \)

From equation 1.25, \( \delta^2 M_x = \frac{W_1 x}{L} (L - x) = \frac{6x}{16} (16 - x) \)

and this is valid from \( x = L - d = 16 - 6 = 10 \) m, to \( x = L = 16 \) m. In this range \( \delta^2 M_x \) is greater than \( 1^1 M_x \) if

\[
-\frac{6x}{16} (16 - x) > (36 - 10x) \left( \frac{16}{16} - x \right)
\]

or if \( 6x > 10x - 36 \) or \( 36 > 4x \) or \( x < 9 \) m.

But since the equation of \( \delta^2 M_x \) is valid for \( x > 10 \) m, the above condition cannot be fulfilled, and hence \( \delta^2 M_x \) is less than \( 1^1 M_x \) between \( x = 10 \) to \( x = 16 \) m. \( \delta^2 M_x \) has its maximum value at \( x = L/2 = 8 \) m, its value being equal to

\[
\delta^2 M_x = +\frac{6 \times 8}{16} (16 - 8) = +24 \text{ kN-m}
\]

The maximum B.M.D. has been drawn in Fig. 1.8(c).

Example 1.3. Solve example 1.2 if \( W_1 = 4 \text{ kN} ; W_2 = 6 \text{ kN} ; d = 6 \text{ m} \) and the span \( L = 12 \text{ m} \).

Solution:

For the present case \( \frac{W_1}{W_1 + W_2} \frac{L}{4 + 6} = \frac{4 \times 12}{4 + 6} = 4.8 \text{ m} \).

\[
\therefore \quad d = 6 \geq \frac{W_1 L}{W_1 + W_2}
\]

Thus the case is not the standard one, and the S.F.D. will not be similar to that of Fig. 1.6(b).

(a) Maximum - ve S.F. Diagram

(i) For \( x = 0 \) to \( x = d = 6 \text{ m} \), when only \( W_1 \) is on the span, and \( W_2 \) is off the span the maximum S.F., from equation 1.12, is given by

\[
1^3 F_{\text{max}} = -R_0 = -\frac{W_1 x}{L} = -\frac{4x}{12} = -\frac{x}{3} \text{ kN}
\]  \( \ldots (1) \)

(ii) For \( x = 0 \) to \( x = 6 \text{ m} \), when \( W_2 \) is just to the left of the section and \( W_1 \) ahead of it.

\[
2^3 F_{\text{max}} = -R_0 + W_1
\]

\[
= -\frac{W_2 x}{L} - \frac{W_1 (x + d)}{L} + W_1 = -\frac{6x}{12} - \frac{4(x + 6)}{12} + 4 = -\left( \frac{6x - 2}{6} \right) \ldots (2)
\]

Thus, for \( x = 0 \) to \( x = 6 \text{ m} \), \( F_{\text{max}} \) is given by both equation (1) and (2). \( 2^3 F_{\text{max}} \) will be greater than \( 1^3 F_{\text{max}} \) only if \( \left( \frac{5x}{6} - 2 \right) \geq \frac{x}{3} \)

or if \( 5x - 12 \geq 2x \)

or if \( 3x > 12 \)

or if \( x > 4 \)

Thus, for \( x = 0 \) to \( x = 4 \text{ m} \), \( 1^3 F_{\text{max}} \) (given by equation 1) gives the maximum S.F. while for \( x = 4 \) to \( x = 6 \text{ m} \), \( 2^3 F_{\text{max}} \) gives the maximum values.
Thus, at \( x = 4 \) m, \( F_4 = -\frac{4}{3} \) kN

at \( x = 6 \) m, \( F_6 = -3 \) kN

(iii) For \( x = d = 6 \) m (or \( L - d = 6 \) m) to \( x = L = 12 \) : both \( W_1 \) and \( W_2 \) on the span:

The maximum S.F. is given by equation 1.13

\[
\begin{align*}
F_{\text{max}} &= - R_B = - \frac{W_1 x + W_2 (x - d)}{L} \\
&= - \frac{4x + 6(x - 6)}{12} = - \frac{5}{6} (x - 3.6) \\
\end{align*}
\]

...(3)

(iv) For \( x = d = 6 \) m to \( x = L = 12 \), with \( W_1 \) on the section and \( W_2 \) off the girder:

The maximum S.F. is given by equation 1.5.

\[
\begin{align*}
F_{\text{max}} &= - R_B = - \frac{W_1 x}{L} = - \frac{6x}{12} = - \frac{x}{2} \\
\end{align*}
\]

...(4)

Thus, for \( x = 6 \) to \( x = 12 \) m, the maximum S.F. is given by equation (3) and (4). Evidently \( F_{\text{max}} \) will be greater than \( F_{\text{max}} \), if

\[
\frac{x}{2} > \frac{5}{6} (x - 3.6) \quad \text{or} \quad x < 9
\]

Thus, from \( x = 6 \) to \( x = 9 \) m, maximum S.F. will be governed by \( F_{\text{max}} \) (equation 4), while from \( x = 9 \) to \( x = 12 \) m, maximum S.F. will be governed by \( F_{\text{max}} \).

At \( x = 9 \) m, \( F_9 = -4.5 \) kN

\( x = 12 \) m, \( F_9 = -7 \) kN

(b) Maximum Positive S.F. Diagram

(i) When both \( W_1 \) and \( W_2 \) are on the girder:

Max. +ve S.F. at \( C \) will occur when \( W_2 \) is just to the right of \( C \) and \( W_1 \) is ahead of it by 6 m.

Thus, from equation 1.18,

\[
\begin{align*}
F_{\text{max}} &= + R_A = + \frac{W_2 (L - x) + W_1 (L - x - d)}{L} \\
&= + \frac{6 (12 - x) + 4 (12 - x - 6)}{12} = + \left( 8 - \frac{5}{6} x \right)
\end{align*}
\]

...(1)

This is valid from \( x = 0 \) to \( x = L - d = 12 - 6 = 6 \) m.

At \( x = 0 \), \( F_{\text{max},\text{max}} = + 8 \) kN; At \( x = 6 \) m, \( F_6 = + (8 - 5) = + 3 \) kN.

(ii) When \( W_2 \) is on the section, and \( W_1 \) off the girder (for \( x = 6 \) m to \( x = 12 \) m).

The max. +ve S.F. given by equation 1.19

\[
\begin{align*}
F_{\text{max}} &= + R_A = + \frac{W_2 (L - x)}{L} = + \frac{6}{12} (12 - x)
\end{align*}
\]

...(ii)

At \( x = 6 \) m, \( F_6 = + \frac{1}{2} (6) = + 3 \) kN, as before,

At \( x = 12 \) m, \( F_9 = 0 \).

The complete S.F.D. is shown in Fig. 1.9(b).

(c) Maximum Bending Moment Diagram:

The maximum bending moment may occur under any one of the following three conditions:
(i) Maximum bending moment under \( W_2 \) with \( W_1 \) ahead of it (\( ^2M_x \)).

(ii) Maximum bending moment under \( W_1 \) and \( W_2 \) behind it (\( ^1M_x \)).

(iii) Maximum bending moment under \( W_2 \) with \( W_1 \) off the span, (\( ^0M_x \)).

Let us investigate all three possibilities:

(i) **Maximum bending moment under \( W_2 \) with \( W_1 \) ahead of it (\( ^2M_x \))**

From equation 1.23,

\[
^2M_x = + R_A \cdot x = + \frac{x}{L} \left[ W_1 (L-x-d) + W_2 (L-x) \right]
\]

\[
= + \frac{x}{12} \left[ 4 (12-x-6) + 6 (12-x) \right] = + x \left[ 8 - \frac{5}{6} x \right] \quad ... (I)
\]

This is zero at \( x = 0 \) and at \( x = \frac{48}{5} = 9.6 \) m

\( ^2M_x \) will have its max. value at \( x = \frac{9.6}{2} = 4.8 \) m

\[
\therefore \quad ^2M_{max} = + 4.8 \left( 8 - \frac{5}{6} \cdot 4.8 \right) = + 19.2 \text{ kN-m}
\]

(ii) **Max. bending moment under \( W_1 \) with \( W_2 \) behind it (\( ^1M_x \))**

From equation 1.22, we have

\[
^1M_x = + \frac{W_1 x + W_2 (x-d)}{L} (L-x) = + \frac{4x + 6 (x-6)}{12} (12-x) = + \left( \frac{5}{6} x - 3 \right) (12-x) \quad ... (II)
\]

This is zero at \( x = 3.6 \) m, and at \( x = 12 \) m

\( ^1M_x \) is maximum at \( x = 3.6 + \frac{12 - 3.6}{2} = 7.8 \) m

\[
\therefore \quad ^1M_{max} = + \left( \frac{5}{6} \cdot 7.8 - 3 \right) (12 - 7.8) = + 14.7 \text{ kN-m}
\]

Now \( ^1M_{max} \) will be greater than \( ^2M_{max} \) for

\[
x > \frac{W_2 L}{W_1 + W_2} > 6 \times \frac{12}{4 + 6} > 7.2 \text{ m}.
\]

Thus for \( x = 0 \) to \( x = 7.2 \) m, max. B.M. will be governed by \( ^2M_{max} \) and from \( x > 7.2 \) m, max. B.M. may be governed by \( ^1M_{max} \).
At $x = 7.2$ m, \[ ^1M_{x2} = + \left( \frac{5}{6} \times 7.2 - 3 \right) (12 - 7.2) = +14.4 \text{ kN-m} \]
\[ ^2M_{x2} = + 7.2 \left( 8 - \frac{5}{6} \times 7.2 \right) = + 14.4 \text{ kN-m}. \] (check)

(iii) **Max. bending moment under $W_1$ with $W_2$ off the girder**

\[ ^0M_x = + \frac{W_1}{L} (L - x) = + \frac{5x}{12} (12 - x) = + x (6 - 0.5x) \] (Eq. 1.25) ...(III)

To get the section where \(^2M_x\) is equal to \(^0M_x\), we have
\[ x \left( 8 - \frac{5}{6} x \right) = x (6 - 0.5x) \quad \text{or} \quad x = 6 \text{ m} \]

The common value of B.M. is given by
\[ ^2M_x = ^0M_x = + 6 (6 - 0.5 \times 6) = + 18 \text{ kN-m} \]

To get the section where \(^0M_x\) and \(^1M_x\) are equal, we have
\[ \left( \frac{5}{6} x - 3 \right) (12 - x) = x (6 - 0.5x), \] which gives $x = 9$ m.

The common value of the max. B.M. is given by
\[ ^0M_x = ^1M_x = + 9 (6 - 0.5 \times 9) = + 13.5 \text{ kN-m} \]

The maximum value of \(^0M_x\) evidently occurs at $x = \frac{L}{2} = 6$ m, its value being equal to $^0M_x = + 6 (6 - 0.5 \times 6) = + 18 \text{ kN-m}$

**Hence, to Summarise:**

(i) For $x = 0$ to $x = 6$ m, Max. B.M. is governed by \(^2M_x\).
(ii) For $x = 6$ m to $x = 9$ m, Max. B.M. is governed by \(^0M_x\).
(iii) For $x = 9$ m to $x = 12$ m, Max. B.M. is governed by \(^1M_x\).

The complete Max. B.M. diagram is shown in Fig. 1.9(c). The absolute Max. B.M. will be under $W_2$, at $x = 4.8$ m, its value being equal to $19.2 \text{ kN-m}$.  

**Example 1.4.** Two point loads, $W_1$ and $W_2$ ($W_2 > W_1$) spaced at a distance ‘d’ travel from left to right across a simply supported girder, with $W_1$ leading. Prove that the limiting span below which the greatest bending moment anywhere in the girder will occur when the load $W_1$ has gone off the girder, is equal to \[ 1 \pm \sqrt{\frac{W_2}{W_1 + W_2}} \] d.

**Hence, draw the max. B.M. diagram if $W_1 = 4 \text{ kN}$ ; $W_2 = 6 \text{ kN}$ ; $d = 6$ m and the span $L = 10$ m.**

**Solution :** (Refer Fig. 1.6 c, d, e).

Since $W_2$ is greater than $W_1$, the maximum value of the B.M. \(^2M_x\) at a section occurs when the load $W_2$ is at the section, with $W_1$ ahead of it. The maximum value of \(^2M_{\text{max}}\) occurs at $x = \frac{AE}{2} = \frac{1}{2} \left( L - \frac{W_1 d}{W_1 + W_2} \right)$. The maximum value of \(^2M_{\text{max}}\) is obtained by substituting the value of $x$ in equation 1.23.
Thus

$$M_{\text{max}} = \frac{3}{L} \left[ W_1 (L - x - d) + W_2 (L - x) \right]$$

$$= \frac{1}{2L} \left[ L - \frac{W_1 d}{W_1 + W_2} \right] \left[ (W_1 + W_2) \left( L - \frac{1}{2} \left( L - \frac{W_1 d}{W_1 + W_2} \right) \right) \right] - W_1 d$$

$$= \frac{1}{4L (W_1 + W_2)} \left[ (W_1 + W_2) L - W_1 d \right] - (W_1 + W_2) L - W_1 d$$

$$= \frac{[(W_1 + W_2) L - W_1 d]^2}{4L (W_1 + W_2)}$$

... (1) ... (1.27)

Also, when $W_1$ is off the girder, the maximum B.M. under $W_2$ at the section is given by equation 1.25,

$$M_x = \frac{W_2 L}{L} \left( L - x \right).$$

Evidently, its maximum value occurs at $x = \frac{L}{2}$

$$M_{\text{max}} = \frac{W_2 L}{L} \left( L - \frac{L}{2} \right) = \frac{W_2 L}{4}$$

... (II)

To have the greatest B.M. governed by $M_{\text{max}}$, we have

$$\frac{W_2 L}{4} > \frac{[(W_1 + W_2) L - W_1 d]^2}{4L (W_1 + W_2)}$$

or

$$W_2 L^2 (W_1 + W_2) > (W_1 + W_2)^2 L^2 + W_1^2 d^2 - 2W_1 dL (W_1 + W_2)$$

or

$$L^2 (W_1 + W_2) [(W_1 + W_2) - W_2] + W_1^2 d^2 - 2W_1 dL (W_1 + W_2) < 0$$

or

$$L^2 (W_1 + W_2) - 2dL (W_1 + W_2) + W_1^2 d^2 < 0$$

which gives

$$L < \left[ 1 \pm \sqrt{1 - \frac{W_1}{W_1 + W_2}} \right] d$$

or

$$L < \left[ 1 \pm \sqrt{\frac{W_2}{W_1 + W_2}} \right] d$$

... (1.28)

**Numerical part:**

To ascertain whether the span is less than that given by equation 1.28, substitute the values in equation 1.28.

Thus,

$$L_{\text{lim}} = \left[ 1 \pm \sqrt{\frac{6}{4 + 6}} \right] \sqrt[6]{1} = 1.35 \text{ m or } 10.65 \text{ m}$$

$$L = 10.65 \text{ m} \text{ (Taking the greater limiting value)}$$

Since our span is even lesser than the greater permissible value, $M_x$ will be greater than $M_{\text{max}}$.

Now

$$M_x = \frac{W_2 x}{L} (L - x) = \frac{6x}{10} (10 - x)$$

This is maximum at $x = \frac{L}{2} = 5 \text{ m}$

Hence

$$M_{\text{max}} = \frac{6 \times 5}{10} \times 5 = 15 \text{ kN\cdotm}.$$
(a) Max. B.M. under $W_2$ with $W_1$ ahead of it ($^3M_x$)  
From Eq. 1.23,  
\[ ^3M_x = + \frac{x}{L} [W_1(L - x - d) + W_2(L - x)] \]
\[ = + \frac{x}{10} [4(10 - x - 6) + 6(10 - x)] = + x(7.6 - x) \]  
...\((i)\)
This is zero at $x = 0$ and $x = 7.6$ m

(b) Max. B.M. under $W_1$ with $W_2$ behind it ($^4M_x$)  
From Eq. 1.23,  
\[ ^4M_x = + \frac{W_1 x + W_2(x - d)}{L} (L - x) \]
\[ = + \frac{4x + 6(x - 6)}{10} (10 - x) \]
\[ = + (x - 3.6)(10 - x) \]  
...\((ii)\)
This is zero at $x = 3.6$ m and $x = 10$ m.

(c) Max. B.M. under $W_2$ with $W_1$ off the girder ($^5M_x$).  
\[ ^5M_x = + \frac{W_2 x}{L} (L - x) \]  
...\((1.25)\)
\[ = + \frac{6x}{10} (10 - x) = + 0.6x(10 - x) \]  
...\((iii)\)
To find the section where $^2M_x$ and $^5M_x$ are equal, we have  
\[ 0.6x(10 - x) = x(7.6 - x) \]
This gives $x = 4$ m.

Thus,  
\[ ^5M_4 = ^5M_4 = + 4(7.6 - 4) = + 14.4 \text{ kN-m}. \]

To find the section where $^4M_x$ and $^5M_x$ will be equal, we have  
\[ 0.6x(10 - x) = (x - 3.6)(10 - x) \]
which gives $x = 9$ m

Thus  
\[ ^1M_9 = ^1M_9 = + 0.6 \times 9(10 - 9) = + 5.4 \text{ kN-m}. \]

Hence, we get

(i) For $x = 0$ to $x = 4$ m, Max. B.M. is governed by $^3M_x$ 
(ii) For $x = 4$ m to $x = 9$ m, Max. B.M. is governed by $^5M_x$. 
(iii) For $x = 9$ m to $x = 10$ m, Max. B.M. is governed by $^4M_x$.

The max. B.M.D. is shown in Fig. 1.10. (b).

**Example 1.5.** Plot the maximum bending moment diagram for a simply supported girder with the following data:

\[ W_1 = 3 \text{ kN (leading)}; \quad W_2 = 6 \text{ kN} \]
\[ d = 6 \text{ m}; \quad L = 10 \text{ m}. \]

Prove that maximum B.M. occurs under $W_2$ when $W_1$ is off the span.
Solution

\[
L_{\text{limit}} = \left(1 \pm \sqrt{\frac{W_2}{W_1 + W_2}}\right) d = \left(1 \pm \sqrt{\frac{6}{3 + 6}}\right) 6 = 10.89 \text{ m.}
\]

Since \(L = 10\) m, maximum B.M. will occur under \(W_2\) when \(W_1\) is off the girder, i.e. \(2M_{\text{max}}\) will be greater than \(2M_{\text{max}}\).

To plot the maximum B.M. diagram, we will investigate all the possibilities.

(a) Maximum B.M. under \(W_2\) with \(W_1\) ahead of it \(\left(2M_x\right)\)

\[
2M_x = \frac{x}{L} \left[ W_1 (L - x - d) + W_2 (L - x) \right] = \frac{x}{10} \left[ 3 (10 - x - 6) + 6 (10 - x) \right] = + x (7.2 - 0.9 x) \quad \text{(1)}
\]

It will be zero at \(x = 0\), and \(x = 8\) m.

(b) Maximum B.M. under \(W_1\) with \(W_2\) behind it \(\left(1M_x\right)\)

\[
1M_x = \frac{W_1 x + W_2 (x - d)}{L} (L - x) = \frac{3 x + 6 (x - 6)}{10} (10 - x) = (0.9 x - 3.6) (10 - x) \quad \text{(2)}
\]

It will be zero at \(x = 4\) m and \(x = 10\) m.

(c) Maximum B.M. under \(W_2\) with \(W_1\) off the girder \(\left(02M_x\right)\)

\[
02M_x = \frac{W_2 x}{L} (L - x) = \frac{6 x}{10} (10 - x) = 0.6 x (10 - x) \quad \text{(3)}
\]

This will be zero at \(x = 0\) m and \(x = 10\) m.

To find the section where \(2M_x\) and \(02M_x\) are equal, we have

\[x(7.2 - 0.9 x) = 0.6 x (10 - x)\]

which gives \(x = 4\) m.

\[
\therefore \quad 2M_4 = 02M_4 = 0.6 \times 4 (10 - 4) = 14.4 \text{ kN-m.}
\]

For the rest of the span, \(02M_x\) will be greater than \(1M_x\) only if

\[0.6 x (10 - x) > (0.9 x - 3.6) (10 - x)\]

or if \[0.6 x > 0.9 x - 3.6\]

or if \[x < 12\] m.

However, since maximum value of \(x = L = 10\) m, \(02M_x\) will always be greater than \(1M_x\).

The absolute maximum B.M. anywhere in the girder will evidently be governed by \(02M_x\). It will occur at \(x = L/2 = 5\) m and its value is

\[M_{\text{max}} = 02M_5 = + 0.6 \times 5 \times 5 = + 15 \text{ kN-m.}\]

The maximum B.M.D. is shown in Fig. 1.11.

![Diagram](image-url)
1.6. SEVERAL POINT LOADS: MAXIMUM B.M.

Let us now take the case of a train of wheel loads \( W_1, W_2 \ldots W_n \) crossing a simply supported girder. For getting the position and amount of maximum bending moment, we shall discuss the following two propositions:

**PROPOSITION 1**

When a series of wheel-loads cross a girder, simply supported at the ends, the maximum bending moment under any given wheel load occurs when the centre of the span is midway between the C.G. of the load system and the wheel load under consideration.

Thus, in Fig. 1.12, let us find the maximum bending moment under the wheel load \( W_3 \), of the train of wheel loads \( W_1, W_2 \ldots W_n \). Let \( W_L \) be the resultant of all loads to the left of \( W_3 \) and \( W_R \) be the resultant of all loads to the right of \( W_3 \) and inclusive of \( W_3 \). Let \( W \) be the resultant of the load system, situated at \( a \) from \( W_L \), \( b \) from \( W_R \) and \( c \) from \( W_3 \). For given load system, \( a, b \) and \( c \) are constants.

To get the maximum B.M. under \( W_3 \), let the load \( W_3 \) be placed at a distance \( z \) from the centre \( C \) of the span. It is required to find the value of the variable \( z \).

Reaction

\[
R_A = \frac{W}{L} \left( \frac{L}{2} + (c-z) \right)
\]

B.M. under \( W_3 \) is

\[
M = R_A \left( \frac{L}{2} + z \right) - W_L (a + c)
\]

\[
= \frac{W}{L} \left( \frac{L}{2} + (c-z) \right) \left( \frac{L}{2} + z \right) - W_L (a + c)
\]

\[
= \frac{W}{L} \left( \frac{L^2}{4} + cz - z^2 + \frac{cL}{2} \right) - W_L (a + c)
\]

For maximum \( M \),

\[
\frac{dM}{dz} = + \frac{W}{L} (c - 2z) = 0 \quad \text{or} \quad z = \frac{c}{2}
\]

Hence the centre of the span is midway between \( W \) and \( W_3 \). This proves the proposition.

The above proposition can be used to find the maximum B.M. under desired wheel load. However, to get absolute maximum B.M. any where on the girder, several trials are to be made. Any one load must first be chosen and arranged according to the condition of equation 1.29 derived above, and the maximum B.M. is calculated. Another wheel load can then be chosen and the procedure repeated to get another value of maximum B.M. Two or three such trials may sometimes be needed, and the absolute maximum B.M. will be the greatest of these. However, to reduce the number of trials to a minimum, the following points must always be kept in mind:

1. The maximum B.M. always occurs under a wheel load, and not any where between two wheel loads.

2. Absolute maximum B.M. always occurs at a section near the centre of the span. (It never occurs at the centre unless the C.G. of the resultant load coincides with the centre line of some heavy wheel load).
3. The wheel load should be so selected that the centre of the span is midway between the C.G. of the load system and wheel load under consideration.

4. The absolute maximum B.M. generally occurs under the heavier wheel load—specially that which is very near to the C.G. of the the load system.

**PROPOSITION 2**

The maximum bending moment at any given section of a simply supported beam, due to given system of point loads crossing the beam occurs when the average loading on the portion to the left of it is equal to the average loading to the right of it, i.e. when the section divides the load in the same ratio as it divides the span.

This proposition is very useful for locating the load position for maximum B.M. at a given section, and has already been proved for uniformly distributed load in § 1.4.

Let it be required to find the load position for maximum B.M. at a point C, distant x from A. Let \( W \) be the resultant load located at y from A, for maximum B.M. at C. Let \( W_L \) be the resultant of the loads to the left of C and \( W_R \) the resultant to the right of C.

\[
R_A = \frac{W(L - y)}{L}
\]

\[
M_X = R_A \cdot x - W_L [x - (y - a)] = W \frac{(L - y)}{L} x - W_L (x - y + a).
\]

In the above expression, y is the only variable. For maximum \( M_X \),

\[
\frac{dM_X}{dy} = - W \frac{x}{L} + W_L = 0 \quad \text{(or should change sign)}
\]

or

\[
\frac{W_L}{x} = \frac{W}{L} = \frac{W - W_L}{L - x} = \frac{W_R}{L - x}
\]

...(1.30)

In other words, the average load on the portion to the left of C is equal to the average load on the portion to the right of C. Actually, in isolated load systems, \( \frac{dM_X}{dy} \) cannot be equal to zero but will have sudden steps changing from a positive value to a negative one. Generally, the loading will be such that either AC is heavier and CB lighter, or vice versa. Hence the maximum B.M. at C will occur when \( \left( \frac{W_L}{x} - \frac{W_R}{L - x} \right) \) changes sign. The value \( \left( \frac{W_L}{x} - \frac{W_R}{L - x} \right) \) can change sign only when a load crosses C from left to right, thus increasing \( W_R \) and decreasing \( W_L \). Hence to get the value of maximum B.M. at a section, one of the wheel loads should be placed at the section, so that if that load is considered as a part of \( W_L \), the expression \( \left( \frac{W_L}{x} - \frac{W_R}{L - x} \right) \) is positive, but if considered as part of \( W_R \), the expression \( \left( \frac{W_L}{x} - \frac{W_R}{L - x} \right) \) becomes negative. If on rolling the loads from left to right, \( \left( \frac{W_L}{x} - \frac{W_R}{L - x} \right) \) does
not change the sign from +ve to −ve, but instead, increases or remains positive, the loads should be rolled to the right so that next load comes over to the section. With this new load at the section, \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) should again be investigated for the two positions, as described above, till it changes sign. In the passage of a series of wheel loads, two or more positions of the load system may occur satisfying the above condition of change of sign from +ve to −ve. In such a case, the value of \( M_x \) at the section for each of these load positions must be calculated, and the greatest of these taken as the maximum B.M. at the section.

It must always be remembered that maximum B.M. at any section occurs when the wheel load is over it.

1.7. SEVERAL POINT LOADS : MAX. S.F. AT A SECTION

Let us now investigate the load position for getting maximum S.F. at a section due to several point loads \( W_1, W_2, \ldots, W_n \). The process of locating the load position for maxima is that of trial and error. However, the max. S.F. at the section occurs when one of the loads is on the section.

To get the max. − ve S.F. at C let the load \( W_1 \) be at the section \( C \), and let another load \( W_2 \) be at \( d \) behind it. If the loads are rolled to the right by a distance \( d \), so that \( W_1 \) comes at \( C \), the S.F. at the section \( C \) will be changed. This change (\( \delta F \)) consists of two components:

(i) Increase \( \delta R_B \) (gradually as the loads roll)

\[
\delta R_B = \frac{Wd}{L}, \quad \text{where} \quad W = \text{resultant of all loads on the span.}
\]

(ii) Sudden decrease or drop equal to \( W_1 \)

Hence

\[
\delta F = \delta R_B - W_1 = \frac{Wd}{L} - W_1 \quad \text{...(1.31)}
\]

If this change is positive, rolling will increase − ve S.F. In such a case, the rolling must be continued till equation 1.31 becomes negative.

The above discussion is true only if no load either enters or leaves the span when the system is rolled by the specified distance \( d \).

To discuss the most general case, let the load \( Q \) enter the span by a distance \( a \), and load \( P \) move beyond \( B \) by a distance \( b \), due to rolling. If \( W \) is the resultant load before the advance, we have

\[
\delta R_B = \frac{Wd}{L} + \frac{Qa}{L} - P \left( 1 + \frac{b}{L} \right)
\]

Hence

\[
\delta F = \delta R_B - W_1 = \frac{Wd}{L} + \frac{Qa}{L} - P \left( 1 + \frac{b}{L} \right) - W_1
\]

\[
= \frac{Wd}{L} - (P + W_1) - \frac{Pb}{L} + \frac{Qa}{L} \quad \text{...(1.32)}
\]
Since \( \frac{b}{L} \) and \( \frac{a}{L} \) are usually small compared with unity, the last two terms of the above expression may be neglected for approximation. Hence, we get

\[
\delta F \approx \frac{W_d}{L} - (W_1 + P)
\]  

... (1.33)

If this is +ve, rolling will increase the S.F. From the above, it is evident that the load entering the span does not change the S.F. appreciably while the load leaving the span does.

If all the loads are equal and equally spaced \( \left( \frac{W_c}{L} - W_1 \right) \) will always be negative and, hence, maximum S.F. at the section will occur when the first load reaches the section.

The absolute maximum -ve S.F. evidently occurs at the right support, for which the criterion of equation 1.33 must be tried.

Equation 1.31 (or 1.33) can also be used for getting maximum +ve S.F. at the section. However, in this case, the advance (or rolling) must be to the left till \( \delta F \) is increased (or becomes +ve). The first chosen wheel load \( W_1 \) is considered just to the right of the section before such movement. If the whole system is now moved to the left, say \( c \), positive S.F. will increase only if \( \left( \frac{W_c}{L} - W_1 \right) \) is positive. If it becomes negative, the load position before such movement gives maximum +ve S.F. If it becomes positive, movement must be permitted till the expression becomes negative.

**Example 1.6.** The system of concentrated loads shown in Fig. 1.15 (a) rolls from left to right across a beam simply supported over a span of 40 m, the 4 kN load leading. For a section 15 m from the left-hand support, determine:

(a) The maximum bending moment.
(b) The maximum shearing force.

**Solution:**

(a) **Maximum B.M.**

By inspection, it is clear that the maximum B.M. at \( C \) will occur when the central 10 kN load is over the section, so that when the loads are rolled across the section, the condition of loading in \( AC - CB \) will alter from heavier-lighter to lighter-heavier. The loads are arranged as shown in Fig. 1.15 (b).

Give small movement to the left,

\[
\frac{W_1}{x} - \frac{W_R}{L-x} = \frac{6 + 6 + 10}{15} - \frac{10 + 4}{40 - 15} = 1.47 - 0.56 = + 0.91.
\]

Giving small movement to the right,

\[
\frac{W_1}{x} - \frac{W_R}{L-x} = \frac{6 + 6 - 10 + 10 + 4}{15} - \frac{10 + 10 + 4}{40 - 15} = 0.8 - 0.96 = - 0.16.
\]
You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
Taking moment of all loads about load no. 5, we get

\[ 92x = (16 \times 3) + (20 \times 6) + (20 \times 10) + (20 \times 14) \]

\[ \therefore \quad x = 7.04 \text{ m} \]

Let us try with the third load (i.e., 20 kN load). Maximum \( M \) under it will occur when the centre of the span is equidistant from load no. 3 and the C.G. of the loads. Distance of load no. 3 and \( W = 7.04 - 6 = 1.04 \text{ m} \).

Distance of load 3 from centre of span

\[ \frac{1.04}{2} = 0.52 \text{ m} \]

i.e. load no. 3 is at a distance of 0.52 m from the centre of the span, as shown in Fig. 1.16 (a). In this position,

\[ R_B = \frac{1}{25} \left[ (16 \times 5.98) + (16 \times 8.98) + (20 \times 11.98) + (20 \times 15.98) + (20 \times 19.98) \right] \]

\[ = 48 \text{ kN} \]

\[ \therefore \quad M_{\text{max}} = + R_B(12.5 + 0.52) - (20 \times 8) - (20 \times 4) = + 48 \times 13.02 - 160 - 80 \]

\[ = +384.96 \text{ kN-m.} \]

(b) Maximum S.F.

Maximum S.F. value is either \( R_A \) or \( R_B \). As the C.G. of the load can approach nearer to \( B \) than to \( A \), \( R_B > R_A \) for limiting load position.

Keep the first load (i.e., 20 kN) just to the left of \( B \). Since next load is at 4 m distance, give a movement of 4 m.

Thus,

\[ W = 92 \text{ kN}; \quad W_1 = 20 \text{ kN} \]

\[ d = 4 \text{ m}; \quad P = \text{load leaving the span} = 20 \text{ kN} \]

From equation 1.33,

\[ \delta F = \frac{Wd}{L} - (W_1 + P) = \frac{92 \times 4}{15} - (20 + 20) = 24.5 - 40 = -15.5. \]

The negative sign shows that the shear force will decrease if the loads are moved. Hence the arrangement of the loads for maximum S.F. will be as shown in Fig. 1.16 (b).

Considering the first 20 kN load just to the left of \( B \), we have

\[ R_B = \frac{1}{25} \left[ (16 \times 11) + (16 \times 14) + (20 \times 17) + (20 \times 21) + (20 \times 25) \right] = 66.4 \text{ kN} \]

\[ \therefore \quad F_{\text{max}} = -R_B = -66.4 \text{ kN}. \]

Example 1.8. A girder, simply supported over a span 20 m, is traversed by moving loads as shown in Fig. 1.17. Determine the maximum B.M at 8 m from the left hand support.

Solution :

Let us try with the second 3 kN load at the section \( C \) as shown.

Giving slight motion to the left:
\[ \frac{W_L - W_R}{x - L - x} = \frac{3 + 3 + 3}{8} - \frac{3 + 8}{12} = 1.125 - 0.91 = +0.215 \]

Giving slight motion to the right:
\[ \frac{W_L - W_R}{x - L - x} = \frac{3 + 3}{8} - \frac{3 + 3 + 8}{12} = 0.75 - 1.117 = -0.42 \]

Since \( \frac{W_L - W_R}{x - L - x} \) changes sign from +ve to -ve, the maximum will occur when the loads are arranged as shown.
\[ R_A = \frac{1}{20} \left[ (8 \times 1 \times 4) + (3 \times 10) + (3 \times 12) + (3 \times 14) + (3 \times 16) \right] = 9.4 \text{ kN} \]

\[ M_{\text{max}} = + (9.4 \times 8) - (3 \times 4) - (3 \times 2) = +57.2 \text{ kN-m} \]

1.8. EQUIVALENT UNIFORMLY DISTRIBUTED LOAD

A given system of loading crossing a girder can always be replaced by uniformly distributed load, longer than the span, such that bending moment or S.F., due to this equivalent static load, every where is atleast equal to that caused by the actual system of moving loads. Such a static load is known as equivalent uniformly distributed load (E.U.D.L). The E.U.D.L. will be different for B.M and S.F. The bending moment diagram for E.U.D.L. will be a parabola symmetrical about the base and must completely envelope the maximum bending moment diagram for the moving loads.

Let us now find the E.U.D.L. for the following cases, for B.M. purposes:
(a) Single point load.
(b) U.D.L. shorter than span.
(c) Two point loads \( W_1 \) and \( W_2 \) at distance \( d \) apart
(a) E.U.D.L. for Single Point Load:
The maximum B.M. at the section \( C \), distant \( x \) from left support due to a single point load is given by equation 1.3,
\[ M_{\text{max}} = + \frac{W \cdot x}{L} (L - x) \]

If \( w' \) is E.U.D.L. over the whole span, B.M. at the section \( C \) is given by
\[ M = + \frac{w' L}{2} - x - \frac{w' x^2}{2} = + \frac{w' x}{2} (L - x) \]

Equating (1) and (2), we get
\[ \frac{w' x}{2} (L - x) = \frac{W \cdot x}{L} (L - x) \]
or
\[ w' = \frac{2W}{L} \]

(1.34)

The same result could be obtained by equating the bending moment at the centre, i.e.,
\[ \frac{w' L^2}{8} = \frac{W L}{4} \]
or
\[ w' = \frac{2W}{L}, \text{ which is the same as above.} \]

(b) E.U.D.L. for U.D.L. shorter than the span:
The max. B.M. at the centre of the span, due to U.D.L. shorter than the span, is given by
\[ M_{\max} = \frac{wL}{4} \left( L - \frac{a}{2} \right) \]
where \( a \) is the length of the U.D.L.
The B.M. at the centre of span, due to E.U.D.L. \( w' \) is
\[ M = \frac{w'L^2}{8} \]
Equating the two, we get
\[ \frac{w'L^2}{8} = \frac{wa}{4} \left( L - \frac{a}{2} \right) \]
or
\[ w' = \frac{2wa}{L^2} \left( L - \frac{a}{2} \right) \] \hspace{1cm} (1.35)

(c) E.U.D.L. for the point loads \( W_1 \) and \( W_2 \) at a distance \( d \) apart:
The E.U.D.L. for this must be such that the B.M.D. due to this completely envelops \( ^2M_x \), \( ^0M_x \), and \( ^1M_x \) diagrams. This can be there if the tangent to the curve of B.M. due to E.U.D.L. at the support is equal to the greater of the tangents to \( ^2M_x \) and \( ^1M_x \) (or \( ^0M_x \)) diagrams at their corresponding ends.

Thus, in Example 1.2, the equation of \( ^2M_x \) is given by
\[ ^2M_x = y = \frac{x}{16} (136 - 10x) \]
\[ \therefore \frac{dy}{dx} \text{ (at } x = 0) = \frac{1}{16} (136) = 8.5 \] \hspace{1cm} (1)
The equation of \( ^1M_x \) is given by
\[ ^1M_x = (10x - 36) \left( 1 - \frac{x}{16} \right) = (12.25x - 0.625x^2 - 36) \]
\[ \therefore \frac{dy}{dx} \text{ (at } x = 16) = 12.25 - 1.25 \times 16 = -7.75 \] \hspace{1cm} (2)
The minus sign simply shows that the inclination of the tangent is in anticlockwise direction.
\[ \therefore \text{ Greater } \frac{dy}{dx} \text{ due to the actual loading } = 8.5 \]
The equation of B.M. at any point, due to E.U.D.L. \( w' \) is
\[ M_x = y = \frac{w'x(L - x)}{2L} = \frac{w'(16 - x)}{2} \]
\[ \therefore \frac{dy}{dx} \text{ (at } x = 0) = 8w' \]
Equating this to the greater of (1) and (2), we get
\[ 8w' = 8.5 \]
\[ w' = \frac{8.5}{8} = 1.06 \text{ kN/m} \]
This will give max. B.M. = \( \frac{w' L^2}{8} = \frac{8.5}{8} \times \frac{16 \times 16}{8} = 34 \) kN-m.

The actual absolute Max. B.M. = 28.8 kN-m, as found in example 1.2.
Similarly, the E.U.D.L. on the considerations of max. shear can also be computed.

1.9. COMBINED DEAD AND MOVING LOAD S.F. DIAGRAMS: FOCAL LENGTH

Let a girder \( AB \), simply supported over a span \( L \), carry a uniformly distributed dead load \( w/\)unit length. Also due to certain system of moving loads, let \( w^* \) be the E.U.D.L., based on shear considerations.

Fig. 1.18(a) shows the S.F.D. due to dead load.

Fig. 1.18(b) shows the S.F.D. due to E.U.D.L. At any distance \( x \), S.F. due to E.U.D.L. is given by

\[
F_x(\text{ve}) = \frac{w^* x^2}{2L} \quad \text{and} \quad F_x(\text{ve}) = \frac{w^* (L - x)^2}{2L}
\]

Fig. 1.18(c) shows the combined S.F.D. obtained after superimposing the two diagrams.

Thus, by combining +ve S.F. of (a) with -ve S.F. of (b), we get final shear = ordinate \( C_1 C_3 \). Similarly by combining +ve S.F. of (a) with +ve S.F. of (b), we get final shear = ordinate \( C_1 C_3 \). Hence in the combined diagram, the final shear at any point is given by vertical intercepts between dead load S.F. and the curves of E.U.D.L.

From Fig. 1.18(c), we make the following observations:

At point \( C \), S.F. = \( C_1 C_2 \) and \( C_1 C_3 \) (both positive)
At point \( P \), S.F. = \( P_1 P_2 (= 0) \) and \( P_1 P_3 \) (positive)
At point \( Q \), S.F. = \( Q_1 Q_2 \) (negative) and \( Q_2 Q_3 (= 0) \).
At point, \( D \), S.F. = \( D_1 D_2 \) and \( D_1 D_3 \) (both negative).

From the above, we make the following conclusions:

(a) For all section to the left of \( P \), the final S.F. is always positive.

(b) For all section to the right of \( Q \), the final S.F. is always negative.

(c) For all section between \( P \) and \( Q \), the final S.F. is both positive and negative. That is, the S.F. changes sign as the load moves over the portion \( PQ \) only. Such a portion of the girder, over which the final S.F. changes sign, is called the focal length. If such a girder is of lattice type, counter bracing is needed for this portion. In Fig. 1.18 (c), thus, \( PQ \) is the focal length of the girder.
Example 1.9. Calculate the focal length of a girder of 16 m span carrying a dead load of 3 kN/m and E.U.D.L. of 6 kN/m for shear.

Solution: (Fig. 1.18)

Let \( F_d \) = S.F. due to dead load, at any section.

\[ F_l = \text{S.F. due to E.U.D.L. at any section.} \]

Then,

\[ F_d = \frac{wL}{2} - wx = \frac{3 \times 16}{2} - 3x = +24 - 3x \quad \ldots (1) \]

\[ F_l (-\text{ve}) = -\frac{wx^2}{2L} = \frac{6x^2}{2 \times 16} = -\frac{3x^2}{16} \quad \ldots (2) \]

At the point \( P \) [Fig. 1.18 (c)], \( F_d + F_l (-\text{ve}) = 0 \)

\[ + 24 - 3x - \frac{3x^2}{16} = 0 \]

which gives \( x = AP = 5.85 \text{ m.} \)

By symmetry, \( BQ = AP = 5.85 \text{ m} \)

\[ \therefore \quad \text{Focal length} = PQ = AB - 2 \times AP = 16 - 2 \times 5.85 = 4.3 \text{ m.} \]

Example 1.10. Calculate the focal length of a girder of 16 m span, carrying a dead load of 3 kN/m and a uniform live load of 2 kN/m, 4 m long, travelling from left to right.

Solution. (Fig. 1.18)

\[ F_d = \frac{wL}{2} - wx = \frac{3 \times 16}{2} - 3x = +24 - 3x \quad \ldots (1) \]

For \( x > 5 \text{ m}, \quad F_l (-\text{ve}) = -\frac{wa}{L} \left( x - \frac{a}{2} \right) \quad \ldots \text{(see equation 1.6)} \]

\[ = -\frac{2 \times 4}{16} \left( x - \frac{4}{2} \right) = -0.5(x - 2) \]

At the point \( P \) [Fig. 1.18 (c)], \( F_d + F_l (-\text{ve}) = 0 \)

\[ + 24 - 3x - 0.5(x - 2) = 0 \]

which gives \( x = AP = 7.14 \text{ m} \)

By symmetry, \( QB = AP = 7.14 \text{ m} \)

\[ \therefore \quad \text{Focal length} = PQ = 16 - 2 \times 7.14 = 1.72 \text{ m.} \]

Example 1.11. Calculate the focal length of the girder of example 1.2 if it also carries a dead load of intensity 3 kN/m over the whole span.

Solution:

For the given girder: \( L = 16 \text{ m}; \quad W_1 = 4 \text{ kN}; \quad W_2 = 6 \text{ kN}; \quad d = 6 \text{ m} \)

For any section distance \( x \) from \( A \),

\[ F_d = +\frac{wL}{2} - wx = \frac{3 \times 16}{2} - 3x = +24 - 3x \quad \ldots (1) \]

(a) For \( x > 6 \), max. -ve S.F. due to live load is given by

\[ F_l (-\text{ve}) = \frac{1}{L} \left( W_1 x + W_2 (x - d) \right) = \frac{1}{L} \left( W_1 + W_2 \right) x - W_2 d \]

\[ = -\frac{(4 + 6)x - 6 \times 6}{16} = -\frac{5x - 18}{8} = -\frac{5}{8}x + 2.25 \quad \ldots (2) \]
At the point $P$ [Fig. 1.18(c)], we have

\[ F_d + F_l (-\text{ve}) = 0 \]

or \[ 24 - 3x - \frac{5}{8}x + 2.25 = 0 \]

which gives \( x = AP = 7.24 \text{ m.} \)

(b) Again, for \( x > 8 < 10 \), we have

\[ F_l (+\text{ve}) = 2F_x = 2 \left[ W_2 (L - x) + W_1 (L - x - d) \right] \]

\[ = + \left[ \frac{6 (16 - x) + 4 (16 - x - 6)}{16} \right] = + 8.5 - \frac{5}{8}x \]

For the point $Q$, we have

\[ F_d + F_l (+\text{ve}) = 0 \]

\[ + 24 - 3x + 8.5 - \frac{5}{8}x = 0 \]

which gives \( x = AQ = 8.96 \text{ m.} \)

Hence focal length = \( AQ - AP = 8.96 - 7.24 = 1.72 \text{ m.} \)

**PROBLEMS**

1. A single rolling load of 10 kN rolls along a girder of 20 m span. Draw the diagrams of maximum B.M. and maximum S.F. positive and negative. What will be the absolute maximum (+) S.F. and B.M.?

2. A uniform load of 1 kN/m, 4 m long crosses a girder of 16 m span. Construct the maximum S.F. and B.M. diagrams and calculate values at section 6 m and 8 m from left hand support.

3. Two concentrated rolling loads of 12 and 6 kN, placed 4.5 m apart, travel along a freely supported girder of 16 m span. Sketch the graphs of maximum shearing force and maximum bending moment and indicate the position and magnitudes of the greater value.

4. A simply supported girder has a span of 40 m. A moving load consisting of a uniformly distributed load of 1 kN/m over a length of 8 m preceded by a concentrated load of 6 kN moving at a fixed distance of 2 m in front of the distributed load, crosses the beam. Find (a) the point of the beam at which the greatest bending moment occurs, (b) the position of the load where it occurs, (c) the value of the greatest B.M.

5. A simply-supported beam is traversed by a train of wheel loads of irregular spacing and unequal weights. State and prove (a) rule giving the train position for the bending moment under a particular load to have its maximum value, and (b) a rule giving the train position for the bending moment at a given point on the beam to its maximum value.

6. A freely supported gantry girder of effective span $L$ carries a travelling crane with two wheel loads, each $W$ at spacing $a$, this spacing being less than $\frac{L}{2}$. Find, from first principles, the maximum bending moment induced by the loads.

If the spacing $a$ is increased, find the maximum value of $a$ (in terms of $L$) for which the maximum bending moment will occur at the centre of the span with only one wheel on the girder. (U.I.)
7. A system of moving loads cross a girder of 36 m span which is simply supported at its ends. The loads and their distances are as follows:

<table>
<thead>
<tr>
<th>Wheel loads (kN)</th>
<th>10</th>
<th>10</th>
<th>20</th>
<th>20</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between centres (m)</td>
<td>3</td>
<td>4.5</td>
<td>4</td>
<td>3.5</td>
<td></td>
</tr>
</tbody>
</table>

Determine
(a) The maximum bending moment at the quarter span.
(b) The maximum bending moment in the girder.

For each case, make a sketch of the girder showing clearly the section where the bending moment occurs and the corresponding position of the loads.

8. The following system of concentrated loads roll from left to right on a span of 15 m, 4 kN load leading:

<table>
<thead>
<tr>
<th>Load</th>
<th>2</th>
<th>6</th>
<th>6</th>
<th>5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (m)</td>
<td>1.5</td>
<td>1.5</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

For a section 4 m from the left hand support, determine (a) the maximum bending moment, (b) maximum S.F.

9. The following system of wheel loads crosses a plate girder of 30 m span:

<table>
<thead>
<tr>
<th>Wheel load</th>
<th>8</th>
<th>18</th>
<th>18</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between centres (m)</td>
<td>4.5</td>
<td>3.5</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Determine the maximum value of the shearing force which may be produced at the middle point of the span. Also, find the equivalent uniformly distributed load which could produce the same maximum bending moment at midspan.

10. A simply supported beam of span \( L \) is crossed by a uniformly distributed load of length \( m \) and of total weight \( W \). If \( L \) is greater than \( m \), obtain from first principles an expression for the maximum bending moment at any point at distance \( a \) from one support. Hence show that a single point load of \( W \left(1 - \frac{m}{2L}\right) \) travelling across the span will give the same maximum moment everywhere along the beam as the above uniformly distributed load. (U.L.)

11. The following arrangement of axle load is carried by a single bridge girder across a clear span of 30 m.

<table>
<thead>
<tr>
<th>Axle loads</th>
<th>5</th>
<th>5</th>
<th>10</th>
<th>10</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spacing (m)</td>
<td>2.5</td>
<td>2.5</td>
<td>2.5</td>
<td>2.5</td>
<td></td>
</tr>
</tbody>
</table>

Determine the maximum bending moment and maximum S.F. at section distant 10 m from left hand abutment. The 5 kN load leads, and the system may pass over the bridge from either side.

12. A beam, simply supported over a span \( L \) is traversed by a uniformly distributed load of intensity \( w \) and length \( \frac{L}{5} \). If the beam also carries a dead load, uniformly distributed over the span, of intensity \( \frac{w}{2} \), indicate on the diagram the length of the beam for which there is reversal of shear force.

ANSWERS

1. S.F. : \( \pm 10 \) kN ; B.M. : \( +50 \) kN-m.

2. 13.12 kN-m ; 14.0 kN-m.
You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
Influence Lines

2.1. DEFINITION

An influence line for any given point or section of a structure is a curve whose ordinates represent to scale the variation of a function, such as shear force, bending moment, deflection, etc. at the point or section as unit load moves across the structure. In other words, an influence line for any given point C on a structure is such a curve that its ordinate at any point D gives the bending moment, shear force or similar quantity at C when a unit load is placed at D. For statically determinate structures, the influence lines for B.M., S.F. or stress are composed of straight lines, while they are curvilinear for statically indeterminate structures. The influence lines are very useful in the speedy determination of the value of a function at the given section under any complex system of loading. These also help to determine, in an easy manner, the disposition of the load system so as to cause the maximum value of the function at the section.

The difference between a curve of B.M. or S.F. (as discussed in the previous chapter) and an influence line of B.M. or S.F. must be clearly understood at this stage. The ordinate of a curve of B.M. or S.F. gives the value of the B.M. or S.F. at the section where the ordinate has been drawn, while in the case of an influence line, the ordinate at any point gives the value of the B.M. or S.F. only at the given section (for which the influence line has been drawn) and not at the point at which the ordinate has been drawn. Also, there is one single B.M. or S.F. curve of the whole beam under the action of a given set or train of loads, while there are infinite number of influence lines, one for each section of the beam, drawn for a unit rolling load.

2.2. INFLUENCE LINE FOR SHEAR FORCE

Let us consider a simply supported beam AB of span L, and construct the influence line for S.F. at section C distant x from the left support. The position of the section is fixed, while the unit load moves from left to right. The problem is to plot the variation of S.F. at the given section C, as the unit load moves along the beam.

At any instant, let the unit load be at a distant \( \alpha L \) from the support A. Then, \( R_B = \alpha \), and \( R_A = (1 - \alpha) \).

\[ \therefore \text{Shear force at } C = F_C = -R_B = -\alpha \]  \hspace{1cm} \text{(1)}

The variation is linear, and is valid for all positions of load between 0 to x from A.
When the load is at \( A \),
\[ \alpha L = 0, \quad \therefore F_C = 0 \]
When the load is at \( C \),
\[ \alpha L = x \quad \text{or} \quad \alpha = x/L \]
\[ \therefore F_C = -\alpha = -x/L \]
When the unit load crosses the section \( C \), \( \alpha > x \), and hence
\[ F_C = + R_A = + (1 - \alpha) \]
Thus, the S.F. changes sign as the unit load crosses the section. The variation is linear, and is valid for all load positions between \( x \) to \( L \) from \( A \).
When the unit load is slightly to the right of \( C \), \( \alpha L = x \quad \text{or} \quad \alpha = x/L \)
\[ \therefore F_C = + (1 - \alpha) \]
\[ = + \left( 1 - \frac{x}{L} \right) \]
\[ = + \frac{L - x}{L} \]
When the unit load is at support \( B \), \( \alpha L = L \) or \( \alpha = 1 \)
\[ \therefore F_C = + (1 - \alpha) \]
\[ = + (1 - 1) = 0 \]

The complete influence line diagram for the S.F. at \( C \) is given in Fig. 2.1 (b).

As per definition, the ordinate \( -y_1 \) at a point gives the S.F. at \( C \), due to unit load at the point where the ordinate \( y_1 \) is measured. Hence if a load \( W_1 \) is acting at that point, and \( y_1 \) is the ordinate of \( I.L. \) under it, the S.F. at \( C = -W_1 y_1 \). Similarly, if a load \( W_2 \) is acting at a certain point, and \( +y_2 \) is the ordinate of the influence line under the point of application of the load, the S.F. at \( C \) will be \( +W_2 y_2 \). If \( W_1 \) and \( W_2 \) are acting simultaneously, the S.F. at \( C = -W_1 y_1 + W_2 y_2 \). Hence if the beam is being acted upon by loads \( W_1, W_2, W_3, ..., W_n \), and \( y_1, y_2, ..., y_n \) are the corresponding influence line ordinate under them, the S.F. at \( C \) is
\[ F_C = W_1 y_1 + W_2 y_2 + W_3 y_3 + ... + W_n y_n = \Sigma W y \]  \hspace{1cm} (2.1)

In the above equation, the numerical value of ordinate \( y \) is to be substituted with its proper algebraic sign, i.e. \(+ve\) if it is of positive diagram and \(-ve\) if of negative portion of the influence line diagram.

Let us now take the case of U.D.L. \((w)\) of length \( a \), placed in the position shown in Fig. 2.1 (c). Let us consider a length \( \delta a \) of the load, and the corresponding elementary load \( \delta W = w \cdot \delta a \). Hence S.F. at \( C \), due to the elementary load \( \delta W \) is
\[ \delta F_C = \delta W \cdot y \]  \hspace{1cm} (where \( y \) is the influence line ordinate under \( \delta W \))
\[ \delta F_c = w \delta a_y \]
\[ = w \times \text{area of the elementary strip of the I.L. diagram} \text{ [shown Fig. 2.1 (d)]} \]

Therefore, the shear force at \( C \), due to total U.D.L. of length \( a \) given by

\[ F_c = \Sigma w (\delta a_y) = w \Sigma \delta a_y \]
\[ = w \times \text{area of I.L. diagram under the U.D.L. [shown shaded in Fig. 2.1 (d)]} \]

Hence the S.F. at \( C \), due to U.D.L. of length \( a \) is equal to the area of the I.L. diagram under the U.D.L. multiplied by the intensity of the load.

Fig. 2.1 (e) shows the U.D.L. extending to both the sides of the section \( C \). In this case, the S.F. at \( C \) is obtained by multiplying the net area by the intensity of the load.

Thus,
\[ F_C = w \left( -a_1 + a_2 \right) \]

where \( a_1 = \text{area of the negative S.F. diagram under the U.D.L.} \)
\( a_2 = \text{area of the positive S.F. diagram under the load.} \)

If \( a_1 = a_2 \), \( F_C = 0 \).

**Influence Line for the Reactions:** If the section \( C \) is located at the support \( B \), the value of \( x = L \) and hence the ordinate of –ve I.L. diagram under \( B = x/L = L/L = 1 \). Thus, the I.L. for reaction at \( B = \text{I.L. for shear at } C \) when \( x = L \), and is a triangle having a maximum ordinate of unity under \( B \). However the I.L. for reactions at \( A \) and \( B \) can be plotted independently as under:

When the load is at a distance \( \alpha L \) from \( A \), \( R_B = \alpha \) and \( R_A = (1 - \alpha) \)

When the load is at \( A \), \( \alpha = 0 \), \( \therefore R_B = 0 \); \( R_A = 1 \)

When the load is at \( B \), \( \alpha = -L \); or \( \alpha = 1 \), \( \therefore R_B = \alpha = 1 \) and \( R_A = (1 - \alpha) = 0 \).

Hence the I.L. for \( R_B \) consists of a triangle having zero ordinate at \( A \) and unit ordinate at \( B \). Similarly, the I.L. for \( R_A \) consists of a triangle having unit ordinate at \( A \) and zero ordinate at \( B \) as shown in Fig. 2.1 (g) and (h) respectively.

2.3. **INFLUENCE LINE FOR BENDING MOMENT**

Let us now construct the I.L. for B.M. at \( C \). When the unit load is at a distance \( \alpha L \) from \( A \), such that \( \alpha L < x \), we have \( R_B = \alpha \) and \( R_A = (1 - \alpha) \).

\[ M_c = + R_B (L - x) \]
\[ = + \alpha (L - x) \]

The variation is linear, and is valid for load position distant 0 to \( x \) from \( A \).

When the unit load is at \( A \),
\[ \alpha L = 0 . \therefore M_c = 0 \]

When the unit load is at \( C \),
\[ \alpha L = x \text{ or } \alpha = x/L \]

\[ M_c = + \frac{x}{L} (L - x) \quad ... (2.3) \]

When the unit load is in \( CB \),

![FIG. 2.2.](image-url)
\[ M_C = + R_A \cdot x = + (1 - \alpha) \cdot x \]  \hspace{1cm} ...(1)

The variation is linear, and is valid for load position distant \( x \) to \( L \) from \( A \).

When the load is at \( C \), \( \alpha L = x \)

\[ M_C = + \left(1 - \frac{x}{L}\right) x = + \frac{L - x}{L} x \]  \hspace{1cm} which is the same as equation 2.3.

Thus, the I.L. diagram for \( M_C \) is a triangle having a maximum ordinate of \( \frac{x}{L} (L - x) \) under the section as shown in Fig. 2.2 (b).

If there are two loads \( W_1 \) and \( W_2 \) acting, and if \( y_1 \) and \( y_2 \) are the influence line ordinates under these loads, we have by definition

\[ M_C = + (W_1 y_1 + W_2 y_2) \]

Hence, if there are number of point loads \( W_1, W_2, \ldots, W_n \) and the corresponding I.L. ordinates under them are \( y_1, y_2, \ldots, y_n \) we have

\[ M_C = + (W_1 y_1 + W_2 y_2 + \ldots \ldots W_n y_n) = + \Sigma W y \]  \hspace{1cm} ...(2.4)

Let there be an U.D.L. of intensity \( w \), and length \( a \), as shown in Fig. 2.2(c). Consider an elementary length \( \delta a \) of the load, such that the elementary load \( \delta W = w \delta a \). Let \( y \) be the average ordinate under the elementary load. Then the B.M. at \( C \) due to this elementary load is given by

\[ \delta W_C = + \delta W \cdot y = + w \delta a \cdot y = + w \times \text{area of the elementary strip of the I.L. diagram} \]

[shown thick in Fig. 2.2(d)].

Hence the B.M. at \( C \), due to the total U.D.L. of length \( a \) is

\[ M_C = + \Sigma w (\delta a \cdot y) = + w \times \text{area of I.L. diagram under U.D.L.} \]

[shown shaded in Fig. 2.2(d)]  \hspace{1cm} ...(2.5)

Thus, the B.M. at \( C \), due to U.D.L. of length \( a \) is equal to the intensity of load multiplied by the area of I.L. diagram under the uniformly distributed load.

### 2.4. LOAD POSITION FOR MAXIMUM S.F. AT A SECTION

In chapter 1 on rolling loads, we have derived the load positions for maximum S.F. at a given section. We will now use the influence line for determination of the position of loads for maximum S.F. at the section \( C \). We shall take different loading conditions.

1. **Single point load.** Let a single point load of magnitude \( W \) roll from left to right. Referring to I.L. of S.F. at the section \( C \) distant \( x \) from \( A \) [Fig. 2.1 (b)] maximum negative S.F. will occur when the load is *just to the left* of \( C \), and maximum +ve S.F. will occur when the load is *just to the right* of \( C \).

Thus,

\[ F_C (\text{max. } -\text{ve}) = - \frac{W \cdot x}{L} \]

and

\[ F_C (\text{max. } +\text{ve}) = + \frac{W (L - x)}{L} \]

2. **U.D.L. longer than the span.** From the I.L. for S.F. at \( C \), Fig. 2.1(b), it is clear that the max. -ve S.F. will occur when the span \( AC \) is loaded and \( CB \) is empty and max. +ve S.F. will occur when the span \( CB \) is loaded and \( AC \) is empty.
Thus, \[ F_C \text{ (max. } -\text{ve}) = w \cdot \frac{1}{2} \frac{x}{L} \cdot \frac{x^2}{2L} \]

and \[ F_C \text{ (max. } +\text{ve}) = w \cdot \frac{1}{2} (L-x) \cdot \frac{(L-x)^2}{2L} \]

3. U.D.L. shorter than the span. Let the U.D.L. of length \( a \) travel from left to right. From Fig. 2.1(b), maximum \(-\)ve S.F. at \( C \) will occur when the head of the load reaches \( C \), while maximum \(+\)ve S.F. will occur when the tail of the load is at \( C \).

4. Several Point Loads. For several point loads, we may use the same criterion, as discussed in the previous chapter. Thus, if a load \( W_1 \) is at the section \( C \), with other loads in appropriate position, and the loads are moved by a distance \( d \) such that next load comes over \( C \), the change \( \delta F_C \) is given by

\[ \delta F_C = \frac{Wd}{L} - W_1. \]

If the above expression is negative, it indicates an increase in S.F. and the loads must be permitted to roll to get greater S.F. The procedure must be repeated till the above expression changes sign, which indicates that greatest peak has been passed.

2.5. LOAD POSITION FOR MAXIMUM B.M. AT A SECTION

Here also, we shall consider all the loading conditions:

1. Single Point Load. Let a single point load \( W \) roll from left to right. Since the I.L. diagram for B.M. at \( C \) has the maximum ordinate under \( C \) itself [see Fig. 2.2 (b)], maximum B.M. will occur when the load is at \( C \) itself.

Thus \[ M_C \text{ (max)} = \frac{W x}{L} (L-x) \]

2. U.D.L. greater than the span. Refer to Fig. 2.2 (b). Maximum B.M. will occur when the U.D.L. occupies the whole span.

Thus \[ M_C \text{ (max)} = w \times \text{ area of I.L. diagram} = w \times \frac{1}{2} \times \frac{L x}{L} (L-x) = \frac{wx (L-x)}{2} \]

3. U.D.L. Shorter than the span. Let the uniformly distributed load be of length \( a \). The load has to be arranged, with respect to section \( C \), in such a way that the area of the I.L. diagram under the load is maximum. Let the load be arranged in the position as shown in Fig. 2.3, so that the shaded area of I.L. diagram is maximum. That is, a small movement of the loading to the left or right will decrease the area of the I.L. diagram. If a movement is given to the left, ordinate \( aa_1 \) will be decreased while \( bb_1 \) will be increased, and the net result will be the decrease in the area of I.L. diagram. Similarly, if a movement is given to the right, ordinate \( bb_1 \) will be decreased while ordinate \( aa_1 \) will be increased and the net result will be the decrease in the area of the I.L. diagram. Evidently, maximum area will be obtained only if the ordinate \( aa_1 \) is equal to ordinate \( bb_1 \).
\[ aa_1 = \frac{x}{L} (L - x) \cdot \frac{AA_1}{x} = \frac{L - x}{L} \cdot AA_1 \quad \text{and} \quad bb_1 = \frac{x}{L} (L - x) \cdot \frac{BB_1}{L - x} = \frac{x}{L} \cdot BB_1 \]

Since \( aa_1 \) must be equal to \( bb_1 \) for maximum area,

\[ \frac{L - x}{L} \cdot AA_1 = \frac{x}{L} \cdot BB_1 \quad \text{or} \quad \frac{x}{L - x} = \frac{AA_1}{BB_1} = \frac{x - AA_1}{L - x - BB_1} = \frac{A_1 C}{CB_1} \]

or

\[ \frac{AC}{CB} = \frac{A_1 C}{CB_1} \quad \text{(2.6)} \]

which is the same as that derived in chapter 1.

Hence the maximum bending moment at a section occurs when the section divides the U.D.L. in the same ratio as it divides the span.

\[ M_C \text{ (max.)} = w \times \text{area of I.L. diagram under the load} \]

\[ = w \left[ \frac{(aa_1 + cc_1)}{2} + \frac{(cc_1 + bb_1)}{2} \right] = w \left( \frac{aa_1 + cc_1}{2} \right) a, \quad \text{since} \quad aa_1 = bb_1 \]

\[ = \frac{w a}{2} \left[ \frac{x(L - x)}{L} \cdot L - a + x(L - x) \right] = \frac{w a x (L - x)}{2 L^2} (2L - a) \quad \text{(2.7)} \]

4. Several Point Loads

Let the loads be so arranged that \( W_L \) is the resultant of the loads to the left of the section \( C \), and \( W_R \) is the resultant of the loads to the right of \( C \). Let \( y_1 \) and \( y_2 \) be the ordinates under \( W_L \) and \( W_R \) respectively.

The B.M. at \( C \), for this arrangement is given by

\[ M_C = W_L \cdot y_1 + W_R \cdot y_2 \]

This is maximum only if a small movement \( \delta d \) of the loads either to the left or to the right, will decrease its value.

Let the loads be given a movement \( \delta d \) to the right, and let the new ordinates under \( W_L \) and \( W_R \) be \( (y_1 + \delta y_1) \) and \( (y_2 - \delta y_2) \) respectively. The corresponding change \( \delta M_C \) is given by

\[ \delta M_C = [W_L (y_1 + \delta y_1) + W_R (y_2 - \delta y_2)] - [W_L \cdot y_1 + W_R \cdot y_2] \]

\[ = W_L \cdot \delta y_1 - W_R \cdot \delta y_2 = W_L \cdot \frac{L - x}{L} \cdot \delta d - W_R \cdot \frac{x}{L} \cdot \delta d \]

\[ = \frac{x (L - x)}{L} \cdot \delta d \left( \frac{W_L}{x} - \frac{W_R}{L - x} \right) \quad \text{(2.9)} \]

Thus, \( \delta M_C \) is negative when \( \frac{W_L}{x} - \frac{W_R}{L - x} \) becomes negative (or changes sign). The value \( \frac{W_L}{x} - \frac{W_R}{L - x} \) can change sign only when a wheel load passes the section \( C \), thus increasing \( W_R \) and decreasing \( W_L \). Thus, to get maximum bending moment at a section, one of the loads should be placed at the section so that if the load is considered as a part of \( W_L \), the expression...
INFLUENCE LINES

\[ \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \] is positive, but if considered as a part of \( W_R \), the expression \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) is negative.

Example 2.1. Two wheel loads of 16 and 8 kN, at a fixed distance apart of 2 m, cross a beam of 10 m span. Draw the influence line for bending moment and shear force for a point 4 m from the left abutment, and find the maximum bending moment and shear force at that point.

Solution (Fig. 2.5).

(a) Max. B.M. at C

The I.L. for B.M. at C distant 4 m from A is shown in Fig. 2.5 (b).

The maximum ordinate under \( C \)

\[ \frac{x (L-x)}{L} = \frac{4 \times 6}{10} = 2.4. \]

The B.M. at \( C \) is maximum when \( \Sigma W_y \) is maximum. By inspection, \( M_{\text{max}} \) occurs when the loads are as in the position shown.

Ordinate under 16 kN load = 2.4

Ordinate under 8 kN load = \( \frac{2.4 \times 4}{6} = 1.6 \)

\[ M_C = (16 \times 2.4) + (8 \times 10.6) = 51.2 \text{ kN.m.} \]

(b) Max. S.F. at C

The I.L. for S.F. at a section \( C \) distant 4 m from A is shown in Fig. 2.5 (c).

The ordinate under \( C \) are \( -\frac{x}{L} = -\frac{4}{10} = -0.4 \) and \( \frac{L-x}{x} = +\frac{6}{10} = 0.6 \).

By inspection of the I.L., max. S.F. occurs when the 16 kN load is just to the right of \( C \), and the 8 kN load is ahead of it. Ordinate under 16 kN load = + 0.6. Ordinate under 8 kN load = + \( \frac{0.6}{6} \times 4 = +0.4 \).

\[ F_C = + \left[ 16 \times 0.6 + 8 \times 0.4 \right] = +12.8 \text{ kN} \]

It can be shown that the max. -ve S.F. at \( C \) will be lesser than 12.8 kN.

Hence maximum S.F. at \( C \) will be 12.8 kN.

Example 2.2 Make neat diagrams of the influence lines for shearing force and B.M. at a section 3 m from one end of a simply supported beam, 12 m long. Use the diagrams to calculate the maximum shearing force and the maximum bending moment at this section due to a uniformly distributed rolling load, 5 m long of 2 kN per meter intensity.

Solution. (Fig. 2.6)

(a) I.L. for B.M.

The ordinate \( c_1 e_2 \) of the I.L. diagram for B.M. at \( C = \frac{3 \times 9}{12} = 2.25 \). The I.L. for B.M. at \( C \) is shown in Fig. 2.6(b). Since the U.D.L. is shorter than the span, the load is to be so arranged that area of I.L. diagram under the load is maximum.
For this condition \(a_1a_2 = b_1b_2\). Let the tail of the load be at a distance \(c\) from \(C\).

From equation 2.6, 
\[
\frac{AC}{CB} = \frac{A_1C}{CB_1} \quad \text{or} \quad \frac{3}{9} = \frac{c}{5-c}
\]

or \(15 - 3c = 9c\)

From which 
\[c = 1.25 \text{ m}\]

The end ordinates are:
\[
a_1a_2 = \frac{2.25}{3} \times 1.75 = 1.3125
\]
\[
b_1b_2 = \frac{2.25}{9} \times 5.25 = 1.3125
\]

\[
M_C = \frac{1.3125 + 2.25}{2} \times (1.25 + 3.75) \times 2 = 17.81 \text{ kN-m}
\]

\((b)\) **I.L. for S.F.**

The I.L. for S.F. at \(C\) is shown in Fig. 2.6(c). The ordinates under \(C\) are

\[-\frac{3}{12} = -0.25\], and \[+\frac{9}{12} = +0.75\]. By inspection, maximum S.F. at \(C\) will occur when the tail of the load is at \(C\). The ordinate under the head of load

\[
= \frac{0.75 \times 4}{9} = \frac{1}{3}
\]

Then 
\[
F_C = w \times \text{(shaded area of I.L. under U.D.L.)} = 2 \times \frac{5}{2} \left(0.75 + \frac{1}{3}\right) = 5.42 \text{ kN}.
\]

**Example 2.3.** A simply supported girder has a span of 25 m. Draw on squared paper the influence line for shearing force at a section 10 m from one end, and using the diagram determine the maximum shearing force due to the passage of a knife-edge load of 5 kN, followed immediately by a uniformly distributed load of 2.4 kN per metre extending over a length of 5 m. The loads may cross in either direction.

**Solution** (Fig. 2.7)

The I.L. ordinate 
\[
cc_1 = -\frac{10}{25} = -\frac{2}{5}
\]
\[
cc_2 = +\frac{15}{25} = +\frac{3}{5}
\]

For maximum -ve S.F., the 5 kN load will be just to the left of \(C\), and the U.D.L. behind or to the left of it. In this position, the ordinate \(aa_1\) under the tail of the U.D.L. is

\[
aa_1 = \frac{2}{5} \times \frac{5}{10} = \frac{1}{5}
\]
\[ F_C(-ve) = - \left( 5 \times \frac{2}{5} \right) - 2.4 \left( \frac{2}{5} + \frac{1}{5} \right) \frac{5}{2} = -5.6 \text{ kN} \]

For maximum +ve S.F. at C, the 5 kN load will be just to the right of C, with U.D.L. to the right of it. In this position, the ordinate \( bb_1 \) under the tail of the load is

\[ bb_1 = \frac{3}{5} \times \frac{10}{15} = \frac{2}{5} \]

\[ \therefore F_C(+ve) = \left( 5 \times \frac{3}{5} \right) + 2.4 \left( \frac{3}{5} + \frac{2}{5} \right) \frac{5}{2} = 9 \text{ kN} \]

Hence the maximum S.F. at the section is the greater of the two. Its value is, therefore, 9 kN.

**Example 2.4.** Four wheel loads of 6, 4, 8 and 5 kN cross a girder of 20 m span, from left to right followed by U.D.L. of 4 kN/m and 4 m long with the 6 kN load leading. The spacing between the loads in the same order are 3 m, 2 m and 2 m. The head of the U.D.L. is at 2 m from the last 5 kN load. Using influence lines, calculate the S.F. and B.M. at a section 8 m from the left support when the 4 kN load is at centre of the span.

**Solution.**

(a) **Bending Moment**

The ordinate of I.L. for B.M. at C

\[ \frac{8 \times 12}{20} = 4.8 \]

When the 4 kN load is at the centre of the beam, the arrangement of the other loads will be as shown in Fig. 2.8 (a).

\[ \therefore \text{Ordinate under } 6 \text{ kN load} = \frac{4.8}{12} \times 7 = 2.8 \]

Ordinate under 4 kN load

\[ \frac{4.8}{12} \times 10 = 4 \]

Ordinate under 5 kN load

\[ \frac{4.8}{8} \times 6 = 3.6 \]

Ordinate under head of U.D.L. = \[ \frac{4.8}{8} \times 4 = 2.4 \]

\[ \therefore M_C = + \sum M_y = + \left[ (5 \times 2.8) + (4 \times 4) + (8 \times 4.8) + (5 \times 3.6) + \left( \frac{1}{2} \times 2.4 \times 4 \times 4 \right) \right] \]

\[ = + 108.4 \text{ kN} \cdot \text{m} \]

\[ \text{...(1)} \]

(b) **Shear Force**

The ordinates of I.L. for S.F. at C are \[ \frac{8}{20} = -\frac{2}{5} = -0.4 \] and \[ + \frac{12}{20} = +\frac{3}{5} = +0.6 \]

Hence ordinate under 6 kN load = \[ \frac{0.6}{12} \times 7 = 0.35 \]

ordinate under 4 kN load = \[ \frac{0.6}{12} \times 10 = 0.5 \]
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When the load is at $B$, \( x = 7 \text{ m}, \) \[ R_B = +1 \]

When the load is at $C$, \( x = 17 \text{ m}, \) \[ R_B = 0. \]

Thus the I.L. for $R_B$ is triangle as shown in Fig. 2.10(c). For maximum $R_B$, let the tail of the U.D.L. be at $x$ from $D$, so that shaded area is maximum. The criterion, given by Eq. 2.6 is

\[
\frac{AD}{DC} = \frac{A_1D}{DC_1} \quad \text{or} \quad \frac{3}{14} = \frac{x}{3-x}, \quad \text{from which} \quad x = \frac{9}{17} \text{ m}
\]

\[
\text{Ordinate } \alpha a_1 = cc_1 = \frac{1.4}{3} \left( 3 - \frac{9}{17} \right) = 1.15
\]

\[
R_B = 2 \times \frac{1}{2} \left[ 1.15 + 1.4 \right] \times 3 = 7.65 \text{ kN } 
\]

(c) I.L. for Reaction at $C$ ($R_C$)

When the unit load is $AD$, distant $x$ from $A$, \( R_A = \left( 1 - \frac{x}{3} \right) \uparrow \) and hence pressure on $DBC$ at $D = \frac{x}{3} \downarrow$

Taking moment about $B$, \( \Sigma M_B = 0 = \frac{x}{3} \times 4 + R_C \times 10 \)

\[
R_C = -\frac{4x}{30} = -\frac{2x}{15} \quad \text{\( i.e. \) } R_C = \frac{2x}{15} \downarrow \quad \ldots (4)
\]

When the load is at $A$, \( x = 0 \) \[ R_C = 0 \]

When the load is at $D$, \( x = 3 \) \text{ m}, \[ R_C = -0.4 \]

Now, let the load be in $DBC$, distant $x$ from $A$. $R_A$ will be zero for the range of load position. Hence taking moments about $B$, we have

\[
\Sigma M_B = 0 = 1 \times (7-x) + R_C \times 10
\]

or

\[
R_C = -\left( 0.7 - \frac{x}{10} \right) \quad \ldots (5)
\]

When the load is at $D$, \( x = 3 \) \text{ m}, \[ R_C = -0.4 \text{ (as before)} \]

When the load is at $B$, \( x = 7 \) \text{ m}, \[ R_C = 0 \]

When the load is at $C$, \( x = 17 \) \text{ m}, \[ R_C = +1 \]

The I.L. for $R_C$ is shown in Fig. 2.10 (d).

By inspection, maximum $R_C$ will occur when the head of the U.D.L. is at $C$. In this position, ordinate under the tail of U.D.L. = $+\frac{1}{10} \times 7 = +0.7$

\[
R_C = \frac{2}{2} \left( 1 + 0.7 \right) 3 = +5.1 \text{ kN (i.e. } 5.1 \text{ kN } \uparrow \)
\]

(d) I.L. for S.F. at a section just to the right of $B$

When the load is between $A$ to $B$, \( F_B = -R_C \), and hence the variation of $F_B$ will be similar to that of $R_C$ but in the reverse direction. Hence I.L. for $F_B$ will have zero ordinate under $A$ and $B$, and ordinate of $+0.4$ under $D$.

When the load is in $BC$, at distance $x$ from $A$, \( R_A = 0 \) and \( R_B = 1.7 - \frac{x}{10} \) (From Eq. 3)
Hence

\[ F_B = + R_B = + \left( 1.7 - \frac{x}{10} \right) \]  

\[ \Rightarrow F_B = + \left( 1.7 - \frac{7}{10} \right) = + 1. 

When the load is just to the right of B, \( x = 7 \) m. \( \therefore F_B = + \left( 1.7 - \frac{7}{10} \right) = + 1. \)

When the load is at C, \( x = 17 \) m

\[ \therefore F_B = + \left( 1.7 - \frac{17}{10} \right) = 0. \]

The complete I.L. diagram is shown in Fig. 2.10 (e). It must be noted that the S.F. is always positive at this section. By inspection, maximum S.F. will occur when the tail of the load is at B. In this position, the ordinate under the head of the U.D.L. is \( \frac{1}{10} \times 7 = 0.7 \).

\[ F_B \text{ (max.)} = + \frac{2}{2} (1 + 0.7) \times 3 = + 5.1 \text{ kN}. \]

(e) I.L. for B.M. at E, 1 m to the right of B

When the unit load is in AD, distant \( x \) from A, \( R_C = \frac{2x}{15} \downarrow \) (from equation 4 above)

\[ M_E = - R_C \times 9 = - \frac{2x}{15} \times 9 = - 1.2x \]  

\[ \therefore M_E = - 1.2x \]  

When the load is A, \( x = 0 \), \( \therefore M_E = 0 \)

When the load is at D, \( x = 3 \) m \( \therefore M_E = - 3.6 \text{ kN.m} \)

When the load is in DE, distant \( x \) from A,

\[ R_C = \left( 0.7 - \frac{x}{10} \right) \downarrow \) (from equation 5 above)

\[ \therefore M_E = - R_C \times 9 = - \left( 0.7 - \frac{x}{10} \right) \times 9 = - 6.3 + 0.9x \]  

\[ \therefore M_E = - 6.3 + 0.9x \]  

When the load is at D, \( x = 3 \) m \( \therefore M_E = - 3.6 \text{ kN.m} \)

When the load is at B, \( x = 7 \) m \( \therefore M_E = 0 \)

When the load is at E, \( x = 8 \) m \( \therefore M_E = + 0.9 \text{ kN.m} \)

When the load is in EC, distant \( x \) from A, \( R_A = 0 \) and

\[ R_B = \left( 1.7 - \frac{x}{10} \right) \uparrow \) (from equation 3 above)

\[ \therefore M_E = + R_B \times 1 = + \left( 1.7 - \frac{x}{10} \right) \]  

When the load is at E, \( x = 8 \) m \( \therefore M_E = + 0.9 \text{ kN.m} \)

When the load is at C \( x = 17 \) m \( \therefore M_E = 0 \)

The complete I.L. for \( M_E \) is shown in Fig. 2.10 (f).

For maximum \( M_E \), let the tail of the U.D.L. be at \( x \) from D. The corresponding area of I.L. diagram is shown shaded. Using criterion of equation 2.6, we have

\[ \frac{AD}{DB} = \frac{A_1 D}{DB_1} \]

or

\[ \frac{3}{4} = \frac{x}{3 - x}, \]  

from which \( x = \frac{9}{7} \) m
\[ \begin{align*}
\therefore \quad \text{Ordinate } aa_1 &= b b_1 = \frac{3.6}{3} \left( 3 - \frac{9}{7} \right) = 2.06 \\
\therefore \quad M_E &= -\frac{2}{2} (2.06 + 3.6)(3) = -16.98 \text{ kN-m} \\
\end{align*} \]

**Example 2.7.** Draw dimensioned influence lines for the reactions at A and C and for the bending moment at E, the mid-point of the lower beam CF of the simply supported beam system shown in Fig. 2.11. By the use of these influence lines, calculate the greatest value of \( R_A, R_C \) and \( M_E \) due to the passage of two 10 kN rolling loads, 2 m apart which travel across the upper beam AB.

**Solution**

(a) *I.L for reaction at A*

Let the unit load be in \( AD \), distant \( x \) from \( A \).

Then \( R_A = \frac{6 - x}{6} = 1 - \frac{x}{6} \) \( \ldots (1) \)

When the load is at \( A \),

\[ x = 0 \]

\[ : \quad R_A = +1 \]

When the load is at \( D \),

\[ x = 6 \text{ m} \]

\[ : \quad R_A = 0 \]

Let the load be in \( DB \), distant \( x \) from \( A \). Then

\[ R_A = -\frac{x - 6}{6} \]

\[ = -\left( \frac{x}{6} - 1 \right) \ldots (2) \]

When the load is at \( D \),

\[ x = 6 \text{ m} \]

\[ : \quad R_A = 0 \]

When the load is at \( B \),

\[ x = 8 \text{ m} \]

\[ : \quad R_A = -\left( \frac{8}{6} - 1 \right) = -\frac{1}{3} \]

The I.L. for \( R_A \) is shown in Fig. 2.11(b).

By inspection, maximum \( R_A \) will be obtained when one 10 kN load is at \( A \) and other ahead of it at 2 m.

\[ : \quad \text{Ordinate under next 10 load } = \frac{1}{6} \times 4 = \frac{2}{3} \]

\[ : \quad R_A = (10 \times 1) + \left( 10 \times \frac{2}{3} \right) = + \frac{50}{3} = +16.67 \text{ kN} \]

(b) *I.L for reaction at C*

When the load is at a distance \( x \) from \( A \),

\[ R_D = \frac{x}{6} \ldots (3) \]

Thus, for the lower beam \( CF \), the downward load at \( D = \frac{x}{6} \)
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3.2. INFLUENCE LINE OF S.F. FOR GIRDER WITH FLOOR BEAMS

Fig. 3.2(a), shows such a system in which the loads are transmitted to the girder at definite points. For a given position of unit load, the S.F. for the whole of the panel, situated between the nodal points, is constant. Hence I.L. is plotted for S.F. of a panel and not for S.F. at a particular section of the girder.

Let such a girder or frame consist of \( n \) panels, each of length \( d \) such that the total length \( L = nd \).

Let us plot the I.L. for S.F. in a panel \( CD \) [Fig. 3.2(b)]. Let there be \( m \) panels to the left of \( CD \), and \( (n - m - 1) \) panels to its right. Thus \( CD \) is \( (m + 1) \)th panel from the left support.

Let the unit load roll from \( A \) to \( G \).

(a) **Load in AC**

When the load is at \( A \), \( R_A = 0 \), \( \therefore F_{CD} = 0 \)

When the load is at \( C \),

\[
R_C = \frac{md}{nd} = \frac{m}{n}
\]

\( \therefore \)

\[
F_{CD} = - R_C = - \frac{m}{n}
\]

(b) **Load in DG**

Now, let the load be in portion \( DG \).

When the load is at \( D \),

\[
R_D = \frac{(n - m - 1)d}{nd} = \frac{n - m - 1}{n}
\]

\( \therefore \)

\[
F_{CD} = + R_D = + \frac{n - m - 1}{n}
\]

When the load is at \( G \),

\[
R_G = 0 \quad \therefore \quad F_{CD} = 0.
\]

(c) **Load in CD**

When the load is at a distance \( x \) from \( C \), load transmitted at the panel point \( C = \frac{d - x}{d} \), and load transmitted at the panel point \( D = \frac{x}{d} \).

\( \therefore \)

\[
F_{CD} = - R_G + \frac{x}{d} = - \frac{md}{nd} - \frac{x}{d}
\]

The variation is linear.

When the load is at \( C \), \( x = 0 \) \( \therefore \quad F_{CD} = - \frac{m}{n} \), as before.
When the load is at $D$, $x = d$. 

Thus the ordinate \( cc_1 = \frac{m}{n} \) and ordinate \( dd_1 = + \frac{n - m - 1}{n} \).

The ordinate is zero at some point \( O \) between \( C \) and \( D \). The I.L. for \( F_{CD} \) is shown in Fig. 3.2(c).

3.3. LOAD POSITIONS FOR MAXIMUM S.F.

Let us now determine the load positions for maximum S.F. in panel \( CD \).

1. **Single Point Load**: Maximum -ve shear will occur when the point load is at \( C \) and maximum +ve shear will occur when the load is at \( D \).

\[
F_{CD} (-\text{ve max.}) = \frac{Wm}{n} \quad \ldots \quad [3.1(a)]
\]

and

\[
F_{CD} (+\text{ve max.}) = \frac{W}{n} (n - m - 1) \quad \ldots \quad [3.1(b)]
\]

2. **U.D.L. greater than the Span**

Maximum -ve S.F. will occur when \( ao \) is fully loaded and \( og \) is empty, and maximum +ve S.F. will occur when \( og \) is fully loaded and \( ao \) is empty, \( o \) being the point of zero ordinate of the I.L. for \( F_{CD} \). The position of the point \( o \) can very easily be located by the consideration of the triangles \( cc_1o \) and \( dd_1o \). Thus,

\[
\frac{cc_1}{dd_1} = \frac{co}{do} = \frac{co}{cd - co}
\]

Since \( cc_1, dd_1, \) and \( cd \) are known, \( co \) can be calculated, and hence \( o \) can be located.

3. **U.D.L. shorter than the span** (Fig. 3.3)

Let a U.D.L. of length \( a \) travel from left to right such that \( a <Ao \), and also \( a <og \). For getting maximum negative shear, the shaded area of the -ve portion of the I.L. should be maximum. For this, the ordinate \( pp_1 \) should be equal to ordinate \( qq_1 \). Hence applying the criterion of equation 2.6,

\[
\frac{AC}{CO} = \frac{PC}{CQ} = \frac{PC}{a - PC} \quad \ldots \quad [3.2(a)]
\]

Since \( AC, CO, \) and \( a \) are known, \( PC \) can be computed, and then the shaded area can be known.

Similarly, for maximum positive shear, the ordinate \( rr_1 \) should be equal to \( ss_1 \). Hence, from criterion of equation 2.6,

\[
\frac{OD}{DG} = \frac{RD}{DS} = \frac{RD}{a - RD} \quad \ldots \quad [3.2(b)]
\]
Knowing OD, DG and a, RD can be computed, and the shaded area can be known.

(4) Irregular Load System

Let a train of wheel loads travel from left to right. Let the arrangement of the load [Fig. 3.2(d)] be such that \( W_1 \) is the resultant of the load to the right of \( CD \), \( W_3 \) is the resultant of the load to the left of \( CD \), and \( W_2 \) is the resultant of the load on the panel \( CD \) itself. This arrangement will give maximum \( F_{CD} \) only if a small movement of the load system decreases the S.F.

Inclination \( \theta \) of \( ac_1 \) or \( gd_1 \) is given by

\[
\tan \theta = \frac{m/n}{md} = \frac{1}{nd} = \frac{1}{L} \]

Inclination \( \phi \) of \( c_1d_1 \) is given by

\[
\tan \phi = \frac{cc_1 + dd_1}{cd} = \frac{(ac + dg)}{cd} \tan \theta = \frac{L - d}{d} \cdot \frac{1}{L} \cdot \frac{L - d}{dL} \]

By giving a small movement \( dx \) to the right, the ordinates \( y_3 \) and \( y_2 \) are increased and ordinate \( y_1 \) is decreased. However, the decrease of \( y_1 \) increases the \( -ve \) S.F. Hence, the change in S.F. is given by

\[
\delta F_{CD} = W_3 \Delta x \tan \theta - W_2 \Delta x \tan \phi + W_1 \Delta x \tan \theta.
\]

\[
\frac{\delta F_{CD}}{dx} = (W_3 + W_1) \tan \theta - W_2 \tan \phi = \frac{W_3 + W_1}{L} - \frac{W_2 (L - d)}{dL}
\]

\[
= \frac{W_1 + W_2 + W_3}{L} - \frac{W}{n} - \frac{W_2}{d} \quad \text{(where \( W \) is the total load)}.
\]

Hence, for the maximum, \( \frac{W}{L} - \frac{W_2}{d} \) (or \( \frac{W}{n} - W_2 \)) should change sign. In the limiting case, when the loads are very near, \( W_2 = \frac{W}{n} \).

Hence the maximum S.F. in a panel occurs when the load in that panel is equal to the load divided by the number of panels.

3.4. INFLUENCE LINE OF B.M. FOR GIRDER WITH FLOOR BEAMS

Let us now draw the I.L. for bending moment at a point \( P \), distant \( x \) from \( A \), in the panel \( CD \).

When the unit load is in \( AC \),

\[ M_p = + R_0 (L - x) \quad \quad \text{...(1)} \]

Again when the unit load is in \( DG \),

\[ M_p = + R_A x \quad \quad \text{...(2)} \]

Both these variations are same as for a girder without floor beam. It must be remembered that \( x \) is a fixed quantity in the above equation.

When the load is at \( C \),

\[ R_0 = \frac{md}{nd} = \frac{m}{n} \]

\[
\therefore \quad M_p = \text{Ordinate } cc_1 = \frac{m}{n} (L - x) \quad \quad \text{...(3)}
\]
When the load is at D,

\[ R_A = \frac{(n-m-1)d}{nd} = \frac{n-m-1}{n} \]  

\[ M_p = \text{Ordinate } dd_1 = \frac{n-m-1}{n} x. \]  

Therefore, if the girder were without the floor beam, the ordinate \( pp_1 \) under the section would have been \( \frac{x}{L} (L-x) \), and the corresponding ordinates \( cc_1 \) and \( dd_1 \) would have been

\[ cc_1 = \frac{x}{L} (L-x) \times \frac{1}{x} \times md \]

\[ = \frac{m}{n} (L-x) \]  

(which is the same as found above in Eq. 3) ; and

\[ dd_1 = \frac{x}{L} (L-x) \times \frac{1}{(L-x)} \times (n-m-1) d \]

\[ = \frac{n-m-1}{n} x \]  

(which is the same as found above in Eq. 4).

Hence the portion \( ac_1 \) and \( bd_1 \) of I.L. for a girder with floor beams can be obtained by constructing the I.L. for the beam assuming it to be without floor beam, making the central ordinate

\[ pp_1 = \frac{x}{L} (L-x) \]  

and joining \( p_1 \) to \( a \) and \( b \), as shown in Fig. 3.4 (b).

To plot the portion of I.L. diagram under the bay \( CD \), consider the unit load at a distance \( a \) from \( C \). The panel point load transferred to \( C \) and \( D \) will be \( \frac{d-a}{d} \) and \( \frac{a}{d} \) respectively. Here

\[ M_p = cc_1 \left( \frac{d-a}{d} \right) + dd_1 \left( \frac{a}{d} \right). \]

This is a linear function of \( a \). When the load is at \( C \), \( a = 0 \) and hence \( M_p = cc_1 \left( \frac{d}{d} \right) = cc_1 \). Similarly, when the load is at \( D \), \( a = d \), and \( M_p = dd_1 \left( \frac{d}{d} \right) = dd_1 \). Hence the I.L. portion under panel \( CD \) is obtained by joining \( c_1 \) and \( d_1 \) by a straight line. The figure \( ac_1 d_1 b \) is thus the complete I.L. diagram for B.M. at point \( P \) is the panel \( CD \).

The I.L. for the point \( P \) in any other panel can also be found in a similar manner. However, when the point \( P \) coincides with some panel or node point, such as \( C \), the I.L. diagram will be a triangle. Fig. 3.4 (d) shows the I.L. for B.M. at \( C \). The ordinate \( cc_1 \) under \( C \) [Fig. 3.4(d)]

\[ = \frac{md}{nd} (n-m)d = \frac{m}{n} (n-m) d. \]

When \( n = 6 \) and \( m = 2 \), \( cc_1 = \frac{2}{6} (6-2) = \frac{4}{3} \).
3.5. LOAD POSITIONS FOR MAXIMUM B.M.

1. Single Point Load

From the inspection of the I.L. for $M_P$, it is clear that $M_{P_{\text{max}}}$ will be obtained by putting the load either at $C$ or at $D$, depending upon whether ordinate $cc_1$ is bigger or $dd_1$ is bigger. Thus

$$M_{P_{\text{max}}} = \frac{Wn}{n} (L - x)$$

or

$$= \frac{W(n - m - 1)}{n} x$$

...3.4 (a)

...3.4 (b)

2. U.D.L. longer than the Span

Maximum B.M. will evidently occur when the load occupies the whole span. In that case, $M_P = w \times \text{shaded area of I.L. diagram}$.

3. Irregular Load System

Let $W_1$ be the resultant of the loads to the right of $D$, $W_3$ the resultant of the loads to the left of $C$, and $W_2$ the resultant of the loads on the panel $CD$. Let the section $P$ be at a distance $b$ from $C$ (such that $b = x - md$). Then, it can be proved that the maximum B.M. at $P$ occurs when the expression

$$\frac{W (L - x)}{L} - \left[ W_2 \left( \frac{d - b}{d} \right) + W_1 \right] \text{ changes sign.}$$

As a rule, $b = \frac{d}{2}$, so that the above criterion reduces to

$$\frac{W (L - x)}{L} - \left( \frac{W_2}{2} + W_1 \right) \text{ changes sign.}$$

...(3.5)

Hence, to get the maximum value of $M_P$, the procedure is as follows: Place the load on the span such that the span is fully covered (if the load system is long) and with one load at $D$. First consider this load as part of $W_2$ and then as part of $W_1$. If this causes the expression of equation 3.5 to change sign, the position is the maximum required. If not, move the load on until another load comes at $C$ or $D$, and apply the above criterion again.

Example 3.1. A Pratt girder shown in Fig. 3.5 consists of eight panels each 3.5 m square, the loading being on the lower boom. Draw the influence line for the member $EC$ and determine the maximum tension and maximum compression in $EC$ due to

(a) a concentrated rolling load of 20 kN.

(b) a uniform live load of 10 kN/m and 10 m long. Indicate clearly for each of the four required values the corresponding load positions.

Solution.

If we pass a section 1.1, it is clear that force in $EC$ is equal to shear in $BC \times \sec 45^\circ = F_{BC} \times \sqrt{2}$.

Hence

$$P_{BC} = F_{BC} \sqrt{2}.$$  

...(1)

Also, when the load is in $ao$, S.F. in panel $BC$ will be negative, and hence force in $EC$ will be compressive. Similarly, when the load is in $od$, S.F. in panel $BC$ will be positive, and hence force in $EC$ will be tensile.

Let us first plot the I.L. for $F_{BC}$.

Here

$m = 2, n = 8; d = 3.5$ m
Ordinate 

\[ bb_1 = \frac{m}{n} = \frac{2}{8} = \frac{1}{4} \]

Ordinate 

\[ cc_1 = \frac{n-m-1}{n} = \frac{8-2-1}{8} = \frac{5}{8} \]

If \( o \) is the point of zero S.F., we have

\[ \frac{bo}{bb_1} = \frac{co}{cc_1} = \frac{bo + co}{bb_1 + cc_1} = \frac{3.5}{\frac{1}{4} + \frac{5}{8}} = 4 \text{ m} \]

\[ \therefore \quad bo = 4 \times bb_1 = 4 \times \frac{1}{4} = 1 \text{ m} \]

and \[ co = 3.5 - 1 = 2.5 \text{ m} \]

(a) Concentrated load of 20 kN

Maximum \(-\)ve S.F. in \( BC \) will occur when the point load is at \( B \), while maximum \(+\)ve S.F. will occur when the point load is at \( C \).

Hence \( F_{BC} \) (\(-\)ve max.) = \( \frac{1}{4} \times 20 = 5 \) kN

\[ \therefore \quad P_{BC} = F_{BC} \sqrt{2} = 5 \sqrt{2} \text{ kN (compressive)} \]

and \( F_{BC} \) (+ve max.) = \( \frac{5}{8} \times 20 = 12.5 \) kN

\[ \therefore \quad P_{BC} = F_{BC} \sqrt{2} = 12.5 \sqrt{2} \text{ kN (tensile)} \]

(b) U.D.L. of 10 kN/m, 10 m long

\[ bo = 1 \text{ m}, \quad \text{found above} \]

\[ \therefore \quad ao = 7 + 1 = 8 \text{ m and } od = 28 - 8 = 20 \text{ m.} \]

Since the length of U.D.L. is more than \( ao \), \(-\)ve S.F. will occur when the load occupies the whole of the portion \( ao \).

Then 

\[ F_{BC} \text{ (\(-\)ve max.)} = w \times \text{Area } ob_1 a \]

\[ = 10 \times \frac{1}{2} \times 8 \times \frac{1}{4} = 10 \text{ kN} \]

\[ \therefore \quad P_{BC} = F_{BC} \sqrt{2} = 10 \sqrt{2} \text{ kN (compressive)} \]

For maximum positive shear, the load should be so arranged that the area of the I.L. diagram under it (shown dotted) is maximum. This will happen when ordinate \( pp_1 = qq_1 \) (i.e. point \( c \) divides the load \( pq \) in the same ratio as it divides the base \( od \) of the triangle \( oc_1 d \)).

Hence applying the criterion of equation 2.6, we have

\[ \frac{oc}{cd} = \frac{pc}{cq} = \frac{pc}{10 - pc} \]

or

\[ \frac{2.5}{20 - 2.5} = \frac{pc}{10 - pc} \]

From which

\[ pc = 1.25 \text{ m} \]

Also,

\[ op = oc - pc = 2.5 - 1.25 = 1.25 \text{ m} \]

Hence

\[ qq_1 = pp_1 = \frac{5}{8} \times \frac{1}{2.5} \times 1.25 = \frac{5}{16} \]

FIG. 3.5.
Hence

\[ F_{BC} \ (\text{+ve max}) = \frac{1}{2} \left( \frac{5}{8} + \frac{5}{16} \right) \times 10 \times 10 = 46.88 \ \text{kN} \]

\[ P_{BC} = F_{BC} \cdot \sqrt{2} = 46.88 \sqrt{2} = 66.2 \ \text{kN} \ (\text{tensile}) \]

PROBLEMS

1. A N-girder bridge (Fig. 3.6) has cross-girders at the lower panel points. The diagonals are at 45°. A live load of 6 kN/m (per girder), longer than the span, crosses the bridge. Find the maximum forces in the three members \( AB, AD, \) and \( CD. \)

\[ \text{FIG. 3.6.} \]

ANSWERS

1. \( P_{AB} = 126 \ \text{kN} \ (\text{com.}) ; \ P_{CD} = 96 \ \text{kN} \ (\text{tensile}) \)

\( P_{AD} = 46.1 \ \text{kN} \ (\text{tension}) \)
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crosses each and every section, and hence the influence line ordinate changes from point to point as the unit load moves along the span. However, in nearly all framed structures and girder with floor beams, the loads (whether knife edge load or uniformly distributed load) are applied at the nodes or joints only, so that we have the case of loads applied at definite points [Fig. 4.1 a,b]. Due to this, the function such as S.F. and B.M. for a panel, or stress in the member of the panel situated between the nodal points, is constant. Hence influence line is plotted for a panel (or for a member), and not for a particular section of a girder or frame.

In this chapter, we shall develop the influence lines for stress (or forces) in the members of the following types of trusses/frames.

1. Pratt truss with parallel chords.
2. Pratt truss with inclined chords.
3. Warren truss with parallel chords.
4. Warren truss with inclined chords.
9. Pannsylvania truss with sub-ties.
10. Pannsylvania truss with sub-struts.
12. Braced cantilever and suspended span girder.

4.2. PRATT TRUSS WITH PARALLEL CHORDS

Fig. 4.2 shows a Pratt truss with 6 panels, each of length 4 m and of height 5 m. Let us draw the influence lines for stresses in members of panels \( L_1 L_2 \) and \( L_2 L_3 \). The truss is statically determinate.

\[
\sin \theta = \frac{5}{\sqrt{4^2 + 5^2}} = \frac{5}{41^{\frac{1}{2}}} = 0.78
\]

\[
\cos \theta = \frac{4}{41} = 0.625 \quad \text{; cosec } \theta = 1.28
\]

(1) Influence line for \( P_{u_1 u_2} \)

In order to find stress \( P_{u_1 u_2} \) in member \( U_1 U_2 \) pass a section \( aa \) as shown. Evidently:

\[
P_{u_1 u_2} = \frac{M_{L_2}}{5} \quad \text{(compression)}
\]

where \( M_{L_2} = \text{bending moment at joint } L_2 \).

The reason for marking the force \( P_{u_1 u_2} \) as compressive becomes quite evident by considering the equilibrium of the portion of the truss to the left of section \( aa \). The left portion of the truss is in equilibrium under the action of external forces (i.e. \( R_4 \) and unit load at \( L_2 \)) and internal forces in members \( U_1 U_2, U_2 L_2 \), and \( L_2 L_3 \). Out of these five forces, the lines of action of forces \( P_{u_2 L_2}, P_{u_2 L_3} \) and the unit load pass through point \( L_2 \) where moments of the forces are being taken. Out of the remaining two forces, moment of \( R_4 \) is clockwise; hence
You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
(3) **Influence line for** $P_{L_1L_2}$

Pass a section $bb$.

\[ P_{L_1L_2} = \frac{M_{U_1}}{5} \text{ (tension)} \]

Hence the I.L. for $P_{L_1L_2}$ will be a triangle, having a maximum ordinate of

\[ \frac{4 \times 20}{24} \times \frac{1}{5} = \frac{2}{3} \text{ under } L_1 \]

as shown in Fig. 4.2 (d).

(4) **Influence line for** $P_{U_1L_1}$

When the unit load is at $L_0$, $P_{U_1L_1} = 0$

When the unit load is at $L_1$, $P_{U_1L_1} = 1$ (tension)

When unit load is at $L_2$ or to the right of $L_2$, $P_{U_1L_1} = 0$

The influence line for $P_{U_1L_1}$ will therefore be a triangle having a maximum ordinate of unity under $L_1$ as shown in Fig. 4.2 (e).

(5) **Influence line for** $P_{L_0U_1}$

The force in $L_0U_1$ can be found by resolution of forces at $A$ in the vertical direction.

When the unit load is at $A$, $R_A = 1$, and hence $P_{L_0U_1} = 0$. When the unit is at $L_1$,

\[ R_A = \frac{20}{24} = \frac{5}{6} \] and

\[ P_{L_0U_1} = R_A \sec \theta = \frac{5}{6} \times 1.28 = 1.07 \text{ (comp.)} \]

When the load is at $B$, $R_A = 0 \therefore P_{L_0U_1} = 0$

The I.L. for $P_{L_0U_1}$ is shown in Fig. 4.2 (f).

**4.3. PRATT TRUSS WITH INCLINED CHORDS**

Fig. 4.3 (c) shows Pratt truss with inclined chords, consisting of 6 panels each of a 4 m length.

(1) **Influence line for** $P_{L_1L_2}$

Pass a section $aa$ cutting three members

\[ P_{L_1L_2} = \frac{M_{U_1}}{U_1L_1} = \frac{M_{U_1}}{3} \text{ (tension)} \]

The influence line diagram will therefore be a triangle having a maximum ordinate

\[ \frac{1}{3} \left( \frac{4 \times 20}{24} \right) = \frac{10}{9} = 1.111 \text{ under } L_1 \text{ as shown in Fig. 4.3 (b).} \]

(2) **Influence line for** $P_{U_1U_2}$

\[ P_{U_1U_2} = \frac{M_{L_2}}{x} \text{ (compression)} \]

where $x$ is the perpendicular distance between point $L_2$ and $U_1U_2$. In order to find $x$, prolong $U_1U_2$ back to meet $L_2L_1$ produced, in $O$.

Now \( \tan \alpha = \frac{5 - 3}{5} = \frac{1}{2} \therefore \alpha = 26^\circ 34' \)

\[ \therefore \sin \alpha = 0.447 \quad \cos \alpha = 0.894 \]
\[ OL_1 = \frac{3}{\tan \alpha} = \frac{3}{\frac{1}{2}} = 6 \text{ m} \]
\[ OL_2 = \frac{5}{\tan \alpha} = \frac{5}{\frac{1}{2}} = 10 \text{ m} \]
\[ OA = 6 - 4 = 2 \text{ m}. \text{ Now,} \]
\[ x = O L_2 \sin \theta = 10 \times 0.447 = 4.47 \text{ m} \]

\[ P_{U_1U_2} = \frac{M_L}{4.47} \]

The influence line diagram will be a triangle having a maximum ordinate
\[ \frac{1}{1.111} \left( \frac{8 \times 16}{24} \right) = 1.193 \]
under \( L_2 \), as shown in Fig. 4.3 (c).

(3) Influence line for \( P_{U_1L_2} \)

\[ P_{U_1L_2} = \frac{M_0}{y} \]

where \( y \) = perpendicular distance of point \( O \) from \( U_1L_2 \)

\[ = O L_2 \sin \theta = 10 \times \frac{3}{5} = 6 \text{ m} \]

\[ P_{U_1L_2} = \frac{M_0}{6} \]

where \( M_0 \) is the moment about \( O \), of the forces to the left of section \( aa \).

When the unit load is at \( A \), \( R_4 = 1 \). Hence considering the forces to the left of the section \( aa \), \( M_0 = 0 \). Hence \( P_{U_1L_2} = 0 \). When the unit load is at \( L_1, R_4 = \frac{20}{24} = \frac{5}{6} \)

\[ P_{U_1L_2} = \frac{M_0}{6} = \frac{1}{6} \left[ (1 \times O L_1) - (R_4 \times A O) \right] = \frac{1}{6} \left\{ (1 \times 6) - \left( \frac{5}{6} \times 2 \right) \right\} = 0.722 \text{ (comp.)} \]

When the unit load is \( L_2 \), \( R_4 = \frac{16}{24} = \frac{2}{3} \)

\[ P_{U_1L_2} = \frac{M_0}{6} = \frac{1}{6} \left( R_4 \times O A \right) \text{ (tension)} = \frac{1}{6} \left( \frac{2}{3} \times 2 \right) = \frac{2}{9} = 0.222 \text{ (tension)} \]

Thus, there is reversal of stress in \( U_1L_2 \) as the load traverses the panel \( L_1L_2 \).

When the load is at \( B \), \( R_4 = 0 \); Hence \( M_0 \) and \( P_{U_1L_2} \) are zero.

The complete I.L. diagram for \( P_{U_1L_2} \) is shown in Fig. 4.3 (d)

(4) Influence line for \( P_{U_2L_2} \)

Pass a section \( bb \) to cut the there members \( U_1U_2, U_2L_2 \) and \( L_2L_3 \). Since \( U_1U_2 \) and \( L_3L_2 \) meet at \( O \), when produced, we have

\[ \text{FIG. 4.3. PRATT TRUSS WITH INCLINED CHORDS.} \]
\[ P_{U2L_2} = \frac{M_0}{O L_2} = \frac{M_0}{10} \]

where \( M_0 \) is the moment, about \( O \), of all forces to the left of section \( bb \).

When the unit load is at \( A \), \( R_A = 1 \).

\[ M_0 = 0 \text{ and hence } P_{U2L_2} \]

When the unit load in between \( A \) and \( L_2 \), \( R_A \) is less than unity, and hence the net moment \( M_0 \) is clockwise. Hence \( P_{U2L_2} \) will give an anti-clockwise moment, giving tensile force in it.

When the load is at \( L_2 \), \( R_A = \frac{16}{24} = \frac{2}{3} \)

\[ P_{U2L_2} = \frac{M_0}{10} = \frac{1}{10} \left\{ (1 \times O L_2) - (R_A \times O A) \right\} \text{ (tension)} \]

\[ = \frac{1}{10} \left\{ (1 \times 10) - \left( \frac{2}{3} \times 2 \right) \right\} = 0.867 \text{ (tension)} \]

When the unit load is at \( L_3 \), \( R_A = \frac{1}{2} \)

\[ P_{U2L_2} = \frac{M_0}{10} = \frac{1}{10} \left\{ R_A \times O A \right\} \text{ (compression)} \]

\[ = \frac{1}{10} \times \frac{1}{2} \times 2 = 0.1 \text{ (compression)} \]

Thus, there is reversal of stress in \( U_2 L_2 \) as the unit load traverses the span \( L_2 L_3 \).

When the unit load is at \( B \), \( R_A = 0 \) and hence \( M_0 \) and \( P_{U2L_2} \) are zero. The I.L. for \( P_{U2L_2} \) is shown in Fig. 4.3 (e).

### 4.4. WARREN TRUSS WITH PARALLEL CHORDS

Fig. 4.4 shows Warren truss with parallel chords, consisting of five panels each of 4 m span. The diagonal members are inclined at 60° to the horizontal. The height \( h \) of the truss is \( 4 \sqrt{3}/2 = 3.4641 \text{ m} \).

\[ \sin 60° = 0.866 \text{ and } \cos 60° = 0.5; \text{ cosec } 60° = 1.1547 \]

1. I.L. for \( P_{U2U_3} \)

Pass a section \( aa \), cutting three members \( U_2 U_3 \), \( U_2 L_2 \) and \( L_1 L_2 \). Evidently,

\[ P_{U2U_3} = \frac{M_{L_2}}{h} = \frac{M_{L_2}}{3.4641} \text{ (compression)} \]

The I.L. diagram will therefore be a triangle having maximum ordinate

\[ = \frac{1}{3.4641} \left( \frac{8 \times 12}{20} \right) = 1.3856 \text{ under point } L_2 \]

The I.L. for \( P_{U2U_3} \) is shown in Fig. 4.4 (b).

2. I.L. for \( P_{L_1L_2} \)

\[ P_{L_1L_2} = \frac{M_{U_2}}{h} = \frac{M_{U_2}}{3.4641} \text{ (tension)} \]

When the load is at \( A \), \( R_A = 0 \) and hence \( M_{U_2} \) and \( P_{L_1L_2} \) are zero.

When the load is at \( L_1 \), \( R_A = \frac{1 \times 4}{20} = \frac{1}{5} \)

\[ P_{L_1L_2} = \frac{M_{U_2}}{3.4641} = \frac{1}{3.4641} \left( \frac{1}{5} \times 14 \right) = 0.8083 \text{ (tension)} \]

When the load is at \( L_2 \), \( R_A = \frac{1 \times 12}{20} = \frac{3}{5} \)

\[ P_{L_1L_2} = \frac{M_{U_2}}{3.4641} = \frac{1}{3.4641} \left( \frac{3}{5} \times 6 \right) = 1.0392 \text{ (tension)} \]
When the load is at $B$, $R_A = 0$ and hence $M_{U_2}$ and $P_{L_1 L_2}$ are zero.

I.L. for $P_{L_1 L_2}$ is shown in Fig. 4.4 (c).

3. I.L. for $P_{U_2 L_2}$

The vertical component of $P_{U_2 L_2}$ is evidently equal to the S.F. in panel $L_1 L_2$. Hence

$$P_{U_2 L_2} = F_{L_1 L_2} \cdot \cosec \theta$$
$$= 1.1547 F_{L_1 L_2}$$

When the load is at $A$, $R_A = 1$ and hence $F_{L_1 L_2}$ as well as $P_{U_2 L_2}$ are each zero. When the load is at $L_1$,

$$P_{U_2 L_2} = 1.1547 \frac{m}{n} = 1.1547 \times \frac{1}{5}$$
$$= 0.2309 \text{ (compressive)}$$

where $m$ is number of panels to the left of $L_1$ and $n$ is the total number of panels. When the load is at $L_2$,

$$P_{U_2 L_2} = 1.1547 \times \frac{n - m - 1}{n}$$
$$= 1.1547 \frac{5 - 1 - 1}{5} = 0.6928 \text{ (tensile)}$$

Fig. 4.4. Warren Truss with Parallel Chords

When the load is at $B$, $R_A = 0$ and hence $F_{L_1 L_2}$ as well as $P_{U_2 L_2}$ are each zero. The I.L. for $P_{U_2 L_2}$ is shown in Fig. 4.4 (d) where we note that stress in member $U_2 L_2$ changes sign as the limit load traverses the panel $L_1 L_2$.

4. I.L. for $P_{U_2 L_1}$

This is also a diagonal member. Pass a section $bb$ as shown in Fig. 4.4 (a). The vertical component of $P_{U_2 L_1}$ is evidently equal to the S.F. in panel $L_1 L_2$.

$$P_{U_2 L_1} = F_{L_1 L_2} \cdot \cosec \theta = 1.547 F_{L_1 L_2}$$

Thus we find that the numerical value of $P_{U_2 L_1}$ is the same as that of $P_{U_2 L_2}$. However, the nature of stress is reversed. When the load is at $L_1$, $P_{U_2 L_1}$ is tensile while when the load is at $L_2$, $P_{U_2 L_1}$ is compressive. The I.L. for $P_{U_2 L_1}$ is shown in Fig. 4.4 (e).

4.5. Waren Truss with Inclined Chords

Fig. 4.5 shows a Warren truss with inclined chords. There are six panels each of 4 m span.

1) Influence line for $P_{U_1 U_2}$

Pass a section $aa$ to cut members $U_1 U_2$, $U_1 L_1$ and $L_1 L_2$.
\[ P_{U1U2} = \frac{M_L}{x} \text{ (compression)} \]

where \( x = \text{perpendicular distance of} \ L_1 \text{ from} \ U_1U_2 \)

\[ = OL_1 \sin \alpha \]

But \( \tan \alpha = \frac{4 - 2}{2} = \frac{1}{2} \)

\[ \therefore \ \alpha = 26^\circ 34' ; \]
\[ \sin \alpha = 0.447 \]

\[ \therefore \ OA = \frac{2}{\tan \alpha} = 2 = 2 \text{ m} \]

\[ \therefore \ OL_1 = 2 + 4 = 6 \text{ m} \]

Hence \( x = OL_1 \sin \alpha \)

\[ = 6 \times 0.447 \]
\[ = 2.68 \text{ m} \]

\[ \therefore \ P_{U1U2} = \frac{M_L}{2.68} \text{ (compression)} \]

The influence line for \( P_{U1U2} \) will be a triangle having a maximum ordinate of \( \frac{4 \times 20}{24} \times \frac{1}{2.68} = 1.24 \) under \( L_1 \), as shown in Fig. 4.5 (b).

\[ (2) \text{ Influence line for} \]
\[ P_{L1L2} \]

\[ P_{L1L2} = \frac{M_U}{4} \text{ (tension)} \]

When the load is at \( A, R_B = 0 \); hence \( M_U \) and \( P_{L1L2} \) are zero.

When the load is at \( L_1, R_B = \frac{4 \times 1}{24} = \frac{1}{6} \)

\[ \therefore \ P_{L1L2} = \frac{M_U}{4} = \frac{1}{4} \left( \frac{1}{6} \times 18 \right) = 0.75 \text{ (tension)} \]

When the load is at \( L_2, R_A = \frac{1 \times 16}{24} = \frac{2}{3} \)

\[ \therefore \ P_{L1L2} = \frac{M_U}{4} = \frac{1}{4} \left( \frac{2}{3} \times 6 \right) = 1 \text{ (tension)} \]

When the load is at \( R_B = 1 \) and \( R_A = 0 \)

Hence \( M_U \) and \( P_{L1L2} \) are zero.

The influence line diagram for \( P_{L1L2} \) has zero ordinates under \( A \) and \( B \) and ordinates of 0.75 and 1.0 under \( L_1 \) and \( L_2 \) as shown in Fig. 4.5 (c).

\[ 3. \text{ Influence line for} \ P_{U1U2} \]

\[ P_{U1U2} = \frac{M_0}{r} \]

where \( r = \text{perpendicular distance of} \ O \text{ from} \ U_2L_1 = OL_1 \sin \beta \)
But

\[ OL_1 = 6 \text{ m} ; \sin \beta = \frac{4}{\sqrt{2^2 + 4^2}} = 0.894 \]

\[ r = 6 \times 0.894 = 5.36 \text{ m} \]

\[ P_{L_1U_2} = \frac{M_0}{5.36} \]

When the unit load is at \( A \), \( R_A = 1 \); Hence \( M_0 \) and \( P_{L_1U_2} \) are zero.

When the unit load is at \( L_1 \), \( R_A = \frac{1 \times 20}{24} = 0.833 \)

\[ P_{L_1U_2} = \frac{1}{5.36} \left( (1 \times 6) - (0.833 \times 2) \right) = 0.808 \text{ (tension)} \]

When the unit load is \( L_2 \), \( R_A = \frac{1 \times 16}{24} = 0.667 \)

\[ P_{L_1U_2} = \frac{1}{5.36} (0.667 \times 2) = 0.248 \text{ (compression)} \]

Thus, there is reversal of stress in \( L_1U_2 \) when the unit load crosses the panel \( L_1L_2 \).

The I.L. for \( P_{L_1U_2} \) is shown in Fig. 4.5 (d).

(4) Influence line for \( P_{U_1U_1} \)

Pass a section \( bb \) to cut members \( U_1U_2, U_1L_1 \) and \( L_0L_1 \)

\[ P_{U_1U_1} = \frac{M_0}{y} \]

where \( y = \) perpendicular distance of \( O \) from \( U_1L_1 = OL_1 \sin \theta \)

\[ OL_1 = 6 \text{ m}; \sin \theta = \frac{2}{\sqrt{2^2 + 2^2}} = \frac{2}{\sqrt{8}} = 0.707 \]

\[ y = 6 \times 0.707 = 4.24 \text{ m} \]

Hence \( P_{U_1U_1} = \frac{M_0}{4.24} \)

When the unit load is at \( A \), \( R_A = 1 \); Hence \( M_0 \) and \( P_{U_1U_1} \) are zero.

When the unit load is \( L_1 \), \( R_A = \frac{20}{24} = 0.833 \)

\[ P_{U_1U_1} = \frac{1}{4.24} \left( 0.833 - 2 \right) = 0.393 \text{ (tension)} \]

When the unit load is at \( B \), \( R_A = 0 \). Hence \( M_0 \) and \( P_{U_1U_1} \) are zero.

The I.L. for \( P_{U_1U_1} \) is shown in Fig. 4.5 (e).

4.6. K-TRUSS

Fig. 4.6 shows K-truss consisting of 8 panels of 4 m each.

(1) Influence line for \( P_{U_2U_3} \)

Pass a section \( aa \) as shown in Fig. 4.6 (a). Considering the equilibrium of the portion to the left of section \( aa \) and taking moments about \( L_2 \), we get

\[ P_{U_2U_3} = \frac{M_{L_2}}{6} \text{ (compression)} \]
The I.L. for $P_{U_{2} U_{3}}$ will be triangle having a maximum ordinate of
\[
\frac{8 \times 24}{32} \times \frac{1}{6} = 1
\]
under $L_2$, as shown in Fig. 4.6 (b).

(2) Influence line for $P_{L_2 L_3}$

\[
P_{L_2 L_3} = \frac{M_{U_2}}{6} \text{ (tension)}
\]

The I.L. for $P_{L_2 L_3}$ will be a triangle having a maximum ordinate of
\[
\frac{8 \times 24}{32} \times \frac{1}{6} = 1
\]
under $L_2$, as shown in Fig. 4.6 (c).

(3) Influence line for $P_{M_2 U_3}$ and $P_{M_2 L_3}$

$M_2 U_3$ and $M_2 L_3$ have the same inclination with the vertical. Hence they will carry equal but opposite stresses. Thus, numerically,

\[
P_{M_2 U_3} = P_{M_2 L_3}
\]

Pass a section $bb$ and consider the equilibrium of the left portion. Resolving vertically,

\[
P_{M_2 U_3} \sin \theta + P_{M_2 L_3} \sin \theta = \text{shear in panel } L_2 L_3 = F_{L_2 L_3}
\]

When the unit load is at $A$, $R_A = 1$, and hence shear in panel $L_2 L_3$ is zero. Therefore, $P_{M_2 U_3}$ and $P_{M_2 L_3}$ are zero. When the unit load is at $L_2$, $R_A = \frac{1 \times 24}{32} = 0.75$

\[
2 P_{M_2 U_3} \sin \theta = F_{L_2 L_3} = 1 - 0.75 = 0.25 \text{ (tension)}
\]

or

\[
P_{M_2 U_3} = \frac{0.25}{2 \sin \theta} \left[ \text{But } \sin \theta = \frac{3}{5} = 0.6 \right]
\]

\[
= \frac{0.25}{2 \times 0.6} = 0.208 \text{ (tension)}
\]

and

\[
P_{M_2 L_3} = 0.208 \text{ (compression)}
\]

FIG. 4.6. K-TRUSS
When the unit load is at $L_3$

$$R_A = \frac{1 \times 20}{32} = \frac{5}{8} = 0.625 = F_{L_2L_3}$$

$$P_{M_2U_3} = \frac{F_{L_2L_3}}{2 \sin \theta} = \frac{0.625}{2 \times 0.6} = 0.52 \text{ (comp.)}$$

and

$$P_{M_2L_3} = 0.52 \text{ (tension)}$$

When the unit load is at $B$, $R_A = 0$. Hence $F_{L_2L_3}$ is zero.

Therefore, $P_{M_2U_3}$ and $P_{M_2L_3}$ are zero.

The I.L. for $P_{M_2U_3}$ and $P_{M_2L_3}$ are shown in Fig. 4.6 (d) and (e) respectively.

(4) Influence line for $P_{M_1L_3}$

Pass a section $cc$, cutting members $U_2U_3$, $M_2U_3$, $M_3L_3$ and $L_3L_4$, and consider the equilibrium of the left portion.

(i) When the unit load is at $L_2$

$$R_A = \frac{1 \times 24}{32} = 0.75 \text{ and } P_{M_2U_3} = 0.208 \text{ (tension)}$$

$$P_{M_2U_3} \sin \theta + P_{M_3L_3} = 1 - R_A = 1 - 0.75 = 0.25$$

$$P_{M_3L_3} = 0.25 - P_{M_2U_3} \sin \theta = 0.25 - (0.208 \times 0.6) = 0.125 \text{ (tension)}$$

(ii) When the unit load is at $L_3$

$$R_A = \frac{1 \times 20}{32} = 0.625 \text{ and } P_{M_2U_3} = 0.52 \text{ (comp.)}$$

$$-P_{M_2U_3} \sin \theta + P_{M_3L_3} = 1 - R_A = 1 - 0.625 = 0.375$$

or

$$P_{M_3L_3} = 0.375 + (0.52 \times 0.6) = 0.687 \text{ (tension)}$$

(iii) When the unit load is at $L_4$

$$R_A = \frac{1 \times 16}{32} = 0.5 \text{ and } P_{M_2U_3} = 0.416 \text{ (comp.)}$$

$$P_{M_3L_3} = R_A - P_{M_2U_3} \sin \theta = 0.5 - (0.416 \times 0.6) = 0.25 \text{ (comp.)}$$

The I.L. for $P_{M_1L_3}$ is shown in Fig. 4.6.

(5) Influence line for $P_{U_3M_3}$

Pass a section $dd$, cutting members $U_3U_4$, $U_3M_3$, $M_1L_3$ and $L_2L_3$. Out of these four, stresses in members $M_3L_3$ and $L_2L_3$ are known. Consider the equilibrium of the portion to the left of section $dd$.

(i) When the unit load is at $L_2$

$$R_A = 0.75 \text{ and } P_{M_2L_3} = 0.208 \text{ (comp.)}$$

$$P_{U_3M_3} = 1 - R_A - P_{M_2L_3} \sin \theta$$

$$= 1 - 0.75 - (0.208 \times 0.6) = 0.125 \text{ (comp.)}$$

(ii) When the unit load is at $L_3$

$$R_A = 0.625 \text{ and } P_{M_2L_3} = 0.52 \text{ (tension)}$$

$$P_{U_3M_3} = R_A - P_{M_2L_3} \sin \theta = 0.625 - (0.52 \times 0.5) = 0.313 \text{ (tension)}$$

The I.L. for $P_{U_3M_3}$ is shown in Fig. 4.6 (g). The influence lines for stresses in other members can similarly be plotted.
4.7. BALTIMORE TRUSS WITH SUB-TIES: THROUGH TYPE

Simple trusses become uneconomical when the span exceeds 80 to 100 m. Earlier, multiple web systems were used in long span bridges. However, they are expensive and highly indeterminate, and are no longer used. The modern trend is to use some form of sub-divided trusses or K-truss. A sub-divided truss is obtained by placing in every panel of the truss some secondary members or diagonals.

In contrast with primary members, which are stressed with all positions of the loads, secondary members are stressed only by the loads in certain limited positions.

Fig. 4.7 (a) shows a Baltimore truss (through type), with subties. The load moves on the lower chords.

(1) Influence line for $P_{L_2 L_3}$ and $P_{L_3 L_4}$

Pass a section $aa$ cutting members $U_2 U_4$, $U_1 M_3$, and $L_2 L_3$.

$P_{L_2 L_3} = P_{L_3 L_4} = \frac{M_{U_2}}{15}$

(tension)

The influence line will be triangle having a maximum ordinate

$12 \times 60 \times \frac{1}{15} = \frac{2}{3}$

under $L_2$, as shown in Fig. 4.7 (b).

(2) Influence line for $P_{U_2 U_4}$

$P_{U_2 U_4} = \frac{M_{L_4}}{15}$

(compression)

When the unit load is at $L_2$

$R_4 = \frac{1 \times 60}{72} = \frac{5}{6}$

FIG. 4.7. BALTIMORE TRUSS WITH SUB-TIES: THROUGH TYPE.
\[ P_{U2}u_4 = \frac{1}{15} \left( \frac{5}{6} \times 24 - 1 \times 12 \right) = \frac{8}{15} = 0.533 \text{ (comp.)} \]

When the unit load is at \( L_3 \),
\[ R_4 = \frac{1 \times 54}{72} = 0.75 \]

\[ P_{U2}u_4 = \frac{1}{15} (0.75 \times 24) = 1.2 \text{ (comp.)} \]

When the load is at \( B \), \( R_4 = 0 \)
\[ P_{U2}u_4 = 0 \]

The influence line for \( P_{U2}u_4 \) is shown in Fig. 4.7 (c).

(3) Influence line for \( P_{U2L2} \)

Pass a section \( bb \). Consider equilibrium of the left portion
\[ P_{U2L2} = \frac{M_a}{12} \text{ (tension)} \]

\( M_a \) and hence \( P_{U2L2} \) are zero when the unit load is at \( A \).

When the unit load is \( L_2 \)
\[ P_{U2L2} = \frac{1}{12} (1 \times 12) = 1 \text{ (tension)} \]

When the unit load is at \( L_1 \) or beyond \( L_1 \) on right side, there is no external force to the left of section \( bb \) except \( R_4 \). Hence \( M_a = 0 \). Therefore, \( P_{U2L2} \) is zero. The I.L. for \( P_{U2L2} \) is, therefore, a triangle having zero ordinates under \( A \) and \( L_3 \) and a maximum ordinate of unity under \( L_2 \), shown in Fig. 4.7 (d).

(4) Influence line for \( P_{U2M3} \)

Considering the equilibrium to the left of section \( aa \),
\[ P_{U2M3} \sin \theta = \text{shear in panel} \ L_2L_3 = F_{L2L3} \]
\[ \sin \theta = \frac{15}{\sqrt{15^2 + 12^2}} = 0.781 \quad \cos \theta = \frac{12}{\sqrt{15^2 + 12^2}} = 0.625 \]

When the unit load is at \( L_2 \), \( R_4 = \frac{1 \times 60}{72} = 0.833 \)
\[ F_{L2L3} = -1 + R_4 = -1 + 0.833 = -0.167 \]
\[ P_{U2M3} = \frac{F_{L2L3}}{\sin \theta} = \frac{0.167}{0.781} = 0.214 \text{ (compression)} \]

When the unit load is at \( L_3 \), \( R_4 = \frac{1 \times 54}{72} = 0.75 = F_{L2L3} \)

\[ P_{U2M3} = \frac{F_{L2L3}}{\sin \theta} = \frac{0.75}{0.781} = 0.96 \text{ (tension)} \]

When the unit load is at \( A \) or \( G \), \( F_{L2L3} \), and \( P_{U2M3} \) are zero. The I.L. for \( P_{U2M3} \) is shown in Fig. 4.7 (e).

(5) Influence line for \( P_{M3L3} \)

\( M_3L_3 \) is a secondary member.

When the unit load is at \( L_2 \) or to the left of \( L_3 \), \( P_{M3L3} = 0 \)
When the unit load is at $L_3$, $P_{M_3L_3} = 1$ (tension)
When the unit load is at $L_4$ or to the right of $L_4$, $P_{M_3L_3} = 0$

The I.L. for $P_{M_3L_3}$ is shown in Fig. 4.7 (f).

(6) **Influence line for** $P_{M_3U_4}$

$M_3U_4$ is a subtie, and is thus a secondary member. Pass a horse shoe section $ee$ cutting five members. Out of these, four members pass through $L_4$ when produced. Hence take the moments about $L_4$. Consider the equilibrium of the portion enclosed by the horse shoe section $ee$.

When the unit load is at $L_2$ or $L_4$, $M_{L_4} = 0$, and hence $P_{M_3U_4} = 0$.

When the unit load is at $L_3$, we get, by taking moment about point $L_4$,

$$P_{M_3U_4} \times (15 \cos \theta) = 1 \times 6 \quad \text{(tension)}$$

$$P_{M_3U_4} = \frac{6}{15 \cos \theta} = \frac{6}{15 \times 0.625} = 0.64 \quad \text{(tension)}$$

Thus, I.L. for $P_{M_3U_4}$ is a triangle, as shown in Fig. 4.7 (g), and is similar to I.L. for $P_{M_3L_3}$. The vertical component of $P_{M_3U_4}$ is equal to $0.64 \sin \theta = 0.64 \times 0.781 = 0.5$. Hence it is very interesting to note that in general, for both parallel and non-parallel chord trusses, where the secondary has the same slope as the main diagonal, the vertical components of stress in the secondary diagonal will be equal to one half of the load applied at the joint.

(6) **Influence line for** $P_{M_3L_4}$

Pass a section $cc$ cutting four members. Consider the equilibrium of the portion to the left of section $cc$.

Resolving forces vertically,

$$P_{M_3U_4} \sin \theta + P_{M_3L_4} \sin \theta = \text{S.F. in panel } L_3 L_4 = F_{L_3L_4}$$

The member $M_3U_4$ will have stress only when the load is in span $L_3L_4$.

(i) When the unit load is at $L_2$, $P_{M_3U_4} = 0$ and $R_4 = 0.833$.

$$F_{L_3L_4} = -1 + 0.883 = -0.167 \quad \text{cosec } \theta = \frac{1}{0.781}$$

$$P_{M_3L_4} = \frac{0.167}{0.781} = 0.214 \quad \text{(compression)}$$

(ii) When the unit load is at $L_3$, $P_{M_3U_4} = 0.64$ (tension)

and

$$R_4 = 0.75$$

$$P_{M_3L_4} = (0.75 - 1) \text{ cosec } \theta + P_{M_3U_4} = -\frac{0.25}{0.781} + 0.64 = 0.32 \quad \text{(tension)}$$

(ii) When the unit load is at $L_4$, $P_{M_3U_4} = 0$ and $R_4 = 0.667$

$$P_{M_3L_4} = 0.667 \text{ cosec } \theta = \frac{0.667}{0.781} = 0.854 \quad \text{(tension)}$$

The I.L. for $P_{M_3L_4}$ is shown in Fig. 4.7 (h).

(7) **Influence line for** $P_{U_4L_4}$

Pass a section $dd$ cutting four sections. Consider equilibrium of left portion. Resolving forces vertically,

$$P_{U_4L_4} + P_{M_3U_4} \sin \theta = \text{shear in panel } L_4 L_5 = F_{L_4L_5}$$

Member $M_3U_4$, has stress when the load is in panel $L_2L_4$ only.
You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
\[ P_{U_2M_3} = 1.28 F_{U_2U_3} \]

(i) When the unit load is at \( U_2 \),
\[ R_4 = 5/6 = 0.833 \]
\[ F_{U_2U_3} = -1 + 0.833 = -0.167 \]
\[ P_{U_2M_3} = 1.28 F_{U_2U_3} = -1.28 \times 0.167 = -0.214 \]
\text{i.e.} \( 0.214 \) (compression)

(ii) When the unit load is at \( U_3 \), \( R_4 = \frac{3}{4} = 0.75 \) and \( F_{U_2U_3} = 0.75 \)
\[ P_{U_2M_3} = 1.28 \times 0.75 = 0.96 \]
\text{(tension)}

Let I.L. for \( P_{U_2M_3} \) is shown in Fig. 4.8 (d).

(4) Influence line for \( P_{U_1M_3} \)

\( U_1 M_3 \) is a secondary member and hence it will be stressed when the load is in panel \( U_2 U_3 \). The influence line for \( P_{U_1M_3} \) is shown in Fig. 4.8 (e), having zero ordinate under \( U_2 \) and \( U_1 \) and an ordinate of unity under \( U_3 \).

(5) Influence line for \( P_{M_3U_4} \)

\( M_3 U_4 \) is a secondary member, and hence it will be stressed only when the load is in panel \( U_2 U_3 \). Pass a horse shoe section \( bb \) cutting four members. Out of these, the line of action of forces in three members pass through point \( U_2 \). Hence take the moment of forces about point \( U_2 \). Consider the equilibrium of the portion enclosed by the section \( bb \).

When the unit load is at \( U_2 \) or \( U_1 \), \( M_{U_2} \) is zero, and hence \( P_{M_3U_4} \) is zero.

When the unit load is at \( U_3 \), we get
\[ M_{U_2} = 1 \times 6 = P_{M_3U_3} \times (12 \sin \theta) \]
\[ P_{M3U4} = \frac{6}{12} \csc \theta = \frac{1}{2} \csc \theta = \frac{1}{2} \times 1.28 = 0.64 \text{ (tension)} \]

The I.L. for \( P_{M3U4} \) is shown in Fig. 4.8 (f).

(6) Influence line for \( P_{M3L4} \)

Pass a section \( cc \) cutting four members \( U3U4, M3U4, M3L4, \) and \( L2L4 \). The stress in \( M3U4 \) is known. Consider the equilibrium of the left portion. Resolving the forces vertically, we get

\[ P_{M3U4} \sin \theta + P_{M3L4} \sin \theta = \text{shear in panel } U3U4 = F_{U3U4} \]

In the above relation, member \( M3U4 \) will have stress only when the load is in \( U2U4 \).

(i) When the unit load is at \( U5 \), \( P_{M3U4} = 0 \) and \( R_A = 5/6 = 0.833 \)

\[ P_{M3L4} = \csc \theta \cdot F_{U3U4} = 1.28 (1 - 0.833) = 0.214 \text{ (comp.)} \]

(ii) When the unit load is at \( U3 \), \( P_{M3U4} = 0.64 \text{ (tension)} \) and \( R_A = 0.75 \)

\[ P_{M3L4} = P_{M3U4} + (0.75 - 1) \csc \theta = 0.64 - 0.25 \times 1.28 = 0.32 \text{ (tension)} \]

(iii) When the unit load is at \( U4 \), \( P_{M3U4} = 0 \) and \( R_A = 2/3 \)

\[ P_{M3L4} = \frac{2}{3} \csc \theta = \frac{2}{3} \times 1.28 = 0.853 \text{ (tension)} \]

The I.L. for \( P_{M3L4} \) is shown in Fig. 4.8 (g). It will be seen that the influence lines for \( P_{U2M3} \) and \( P_{M3L4} \) are exactly the same for load position between \( U3 \) to \( U2 \) and between \( U4 \) to \( U12 \).

(7) Influence line for \( P_{U4L4} \)

Pass a section \( dd \), cutting four members. Considering the equilibrium of the left portion and resolving the forces vertically we get, in general.

\[ P_{U4L4} + P_{M3U4} \sin \theta = \text{shear in panel } U3U4 = F_{U3U4} \]

Member \( M3U4 \) will have stress only when the load is in \( U2U4 \).

(i) When the unit load is at \( U5 \), \( R_A = 0.833 \) and \( P_{M3U2} = 0 \)

\[ F_{U4L4} = 1 - 0.833 = 0.167 \text{ (tension)} \]

(ii) When the unit load is at \( U3 \), \( R_A = 0.75 \)

\[ P_{M3U4} = 0.64 \text{ (tension)} \]

\[ P_{U4L4} = (0.75 - 1) + (0.64 \times 0.781) \text{ (compression)} = -0.25 + 0.5 = 0.25 \text{ (compression)} \]

(iii) When the unit load is at \( U4 \), \( P_{M3U4} = 0 \) and \( R_A = \frac{2}{3} \)

\[ P_{U4L4} = \frac{2}{3} = 0.667 \text{ (compression)} \]

The I.L. for \( P_{U4L4} \) is shown in Fig. 4.8 (h).

4.9. BALTIMORE TRUSS WITH SUB-STRUTS : THROUGH TYPE

(1) Influence line for \( P_{U2U4} \)

Pass a section \( aa \) cutting three members. Considering equilibrium of the left portion.

\[ P_{U2U4} = \frac{M_{L4}}{15} \text{ (compression)} \]
The influence line diagram for \( P_{U_2L_4} \) is thus a triangle having a maximum ordinate of
\[
\frac{24 \times 48}{72} \times \frac{1}{15} = 1.067 \text{ (comp.) under } L_4 \text{ as in Fig. 4.9 (b).}
\]

(2) Influence line for \( P_{L_2L_4} \)

\[
P_{L_2L_4} = \frac{M_{U_2}}{15} \text{ (tension)}
\]

When the unit load is at \( L_2 \) , \( R_A = \frac{60}{72} = 0.833 \)

\[
P_{L_2L_4} = \frac{1}{15} (0.833 \times 12) = 0.667 \text{ (tension)}
\]

When the unit load is at \( L_3 \) , \( R_A = \frac{54}{72} = 0.75 \)

\[
P_{L_2L_4} = \frac{1}{15} (0.75 \times 12 + 1 \times 6) = 1 \text{ (tension)}
\]

When the unit load is at \( L_4 \) , \( R_A = \frac{48}{72} = 0.667 \)

\[
P_{L_2L_4} = \frac{1}{15} (0.667 \times 12) = 0.533 \text{ (tension)}
\]

The I.L. diagram for \( P_{L_2L_4} \) is shown in Fig. 4.9 (c).

(3) Influence line for \( P_{M_3L_4} \)

Pass section \( aa \). Resolving the forces vertically,

\[
P_{M_3L_4} \sin \theta = \text{shear in panel } L_3L_4 = F_{I_3L_4}
\]

or

\[
P_{M_3L_4} = F_{I_3L_4} \csc \theta
\]

\[
\sin \theta = \frac{15}{\sqrt{15^2 + 12^2}} = 0.781; \csc \theta = 1.28; \cos \theta = 0.625
\]

When the unit load is at \( L_3 \) , \( R_A = 0.75 \)

\[
P_{M_3L_4} = (1 - 0.75) \times 1.28 = 0.32 \text{ (comp.)}
\]

When the unit load is at \( L_4 \) , \( R_A = 0.667 \)

\[
P_{M_3L_4} = 0.667 \times 1.28 = 0.853 \text{ (tension)}
\]

The I.L. diagram for \( P_{M_3L_4} \) is shown in Fig. 4.9 (d).

(4) Influence line for \( P_{M_3L_3} \)

\( M_3L_3 \) is a secondary member and hence it will be stressed only when the load is in panel \( L_2L_4 \) will be a Evidently, I.L. for \( P_{M_3L_3} \) will be a triangle having zero ordinates under \( L_2 \) and \( L_3 \) and an ordinate of unity under \( L_3 \) as shown in Fig. 4.9 (e).

(5) Influence line for \( P_{L_2M_3} \)

Pass a horse shoe section \( bb \), cutting five members \( u_2M_3, M_3L_4, L_2M_3, L_2L_3 \) and \( L_3L_4 \). Out of these, line of action of forces in four members pass through point \( L_4 \). Hence take the moment of the forces, about \( L_4 \) and consider the equilibrium of the portion enclosed by the horse shoe section. Note that \( L_2M_3 \) is a secondary member, and hence it will be stressed only when the load is in panel \( L_2L_4 \).

When the unit load is at \( L_2 \), \( M_{L_4} = 0 \) and hence \( P_{L_2M_3} = 0 \).
When the unit load is at $L_3$,

$$M_{L4} = 1 \times 6 = P_{L2M3} \times (12 \sin \theta)$$

$$P_{L2M3} = \frac{6}{12 \sin \theta} = \frac{1}{2} \csc \theta = \frac{1}{2} \times 1.28 = 0.64 \text{ (comp.)}$$

When the unit load is at $L_4$, $M_{L4} = 0$, and hence $P_{L2M3} = 0$

The I.L. diagram for $P_{L2M3}$ is shown in Fig. 4.9 (f).

(6) Influence line for $P_{U2M3}$

Pass a section cc, cutting four members. Considering the equilibrium of the left portion, we get, in general,

$$P_{U2M3} \sin \theta + P_{L2M3} \sin \theta = \text{shear in } L_2 L_3 = F_{L2L3}$$

or

$$P_{U2M3} + P_{L2M3} = L_2 L_3 \csc \theta = 1.28 F_{L2L3}$$

The member $L_2 M_3$ will be stressed only when the load is in $U_2 U_4$. When the unit load is at $L_2$, $R_4 = 0.833$ and $P_{L2M3} = 0$. Hence,

$$P_{U2M3} = (1 - 0.833) \times 1.28 = 0.214 \text{ (comp.)}$$

When the unit load is at $L_3$, $R_4 = 0.75$ and $P_{L2M3} = 0.64$ (comp.)

$$P_{U2M3} + 0.64 = 0.75 \times 1.28$$

or

$$P_{U2M3} = 0.75 \times 1.28 - 0.64 = 0.32 \text{ (tension)}$$

When the unit load is at $L_4$, $R_4 = 0.667$ and $P_{L2M3} = 0$. Hence,

$$P_{U2M3} = 0.667 \times 1.28 = 0.835 \text{ (tension)}$$

The I.L. diagram for $P_{U2M3}$ is shown in Fig. 4.9 (g).

\[\begin{figure}
\text{FIG. 49. BALTIMORE TRUSS WITH SUB-STRUTS: THROUGH TYPE}
\end{figure} \]
(7) Influence line for \( P_{U2L2} \)

Pass a section \( dd \), cutting four members \( U_2 M_1, U_2 L_2, L_2 M_3 \) and \( L_2 L_3 \). Out of these, the lines of action of forces in two members pass through \( A \). Hence take the moment of all the forces about \( A \) and consider the equilibrium of the left portion.

When the unit load is at \( A \), \( P_{U2L2} \) is evidently zero.

When the unit load is at \( L_2 \), \( P_{L2L2} = 0 \)

\[
P_{U2L2} \times 12 = 1 \times 12
\]

or

\[
P_{U2L2} = 1 \text{ (tension)}
\]

When the unit load is at \( L_3 \), \( P_{L2M3} = 0.64 \text{ (comp.)} \)

\[
P_{U2L2} \times 12 = (P_{L2M3} \sin \theta) \times 12
\]

\[
P_{U2L2} = 0.64 \times 0.781 = 0.5 \text{ (tension)}
\]

When the unit load is at \( A \), \( P_{L2M3} = 0 \)

\[
P_{U2L2} \times 12 = \text{zero}
\]

The I.L. for \( P_{U2L2} \) is shown in Fig. 4.9 (h).

4.10. PENNSYLVANIA OR PETTIT TRUSS WITH SUB-TIES

(1) Influence line for \( P_{U4U6} \)

Pass a section \( aa \) cutting three members

\[
P_{U4U6} = \frac{M_{L6}}{r} \text{ (compression)}
\]

where \( r = \) perpendicular distance of \( L_6 \) from \( U_4 U_6 \).

Prolong \( U_6 U_4 \) backward to meet \( BA \) produced in \( O_1 \). Let \( \theta_1 \) be the inclination of \( U_4 U_6 \) as shown in Fig. 4.10 (a).

\[
\tan \theta_1 = \frac{15 - 13}{12} = \frac{1}{6} = 0.1667 \; ; \; \theta_1 = 9^\circ 28'
\]

\[
\sin \theta_1 = 0.165 \; ; \; \cos \theta_1 = 0.986
\]

\[
O_1 L_6 = \frac{U_6 L_6}{\tan \theta_1} = 15 \times 6 = 90; \; ; O_1 A = 90 - 36 = 54 \; \text{m}
\]

Now

\[
r = O_1 L_6 \sin \theta_1 = 90 \times 0.165 = 14.85 \; \text{m}
\]

\[
\therefore \quad P_{U4U6} = \frac{M_{L6}}{14.85} \text{ (compression)}
\]

When the unit load is at \( L_4 \), \( R_4 = \frac{48}{72} = 0.667 \)

\[
\therefore \quad P_{U4U6} = \frac{1}{14.85} [1.667 \times 36 - 1 \times 12] = 0.808 \text{ (comp.)}
\]

When the unit load is at \( L_5 \), \( R_4 = \frac{42}{72} = 0.583 \)

\[
\therefore \quad P_{U4U6} = \frac{1}{14.85} [0.583 \times 36] = 1.412 \text{ (comp.)}
\]

The I.L. diagram for \( P_{U4U6} \) is shown in Fig. 4.10 (b).

(2) Influence line for \( P_{L4L6} \)

\[
P_{L4L6} = \frac{M_{L4}}{13} \text{ (tension)}
\]
The I.L. diagram for $P_{u4}L_4$ will, therefore, be a triangle having a maximum ordinate of 
\[
\frac{24 \times 48}{72} \times \frac{1}{13} = 1.23
\]
under $L_4$ as shown in Fig. 4.10 (c).

(3) Influence line for $P_{u4}M_5$

Consider the equilibrium of the portion to the left of section aa. Take moment about $O_1$, where two members $U_4U_6$ and $L_4L_6$ when produced.

\[
P_{u4M_5} = \frac{M_{O_1}}{\bar{x}}
\]

where $\bar{x}$ = perpendicular distance of $U_4M_5$ from $O_1 = O_1L_6\sin \theta$

$\theta$ = inclination of $U_4L_6$ with $O_1L_6$

Now

\[
\sin \theta = \frac{13}{\sqrt{13^2 + 12^2}} = 0.734
\]

$\cos \theta = 0.678$

$\csc \theta = 1.364$

\[
\bar{x} = 90 \times 0.734 = 66.1 \text{ m}
\]

\[
P_{u4M_5} = \frac{M_{O_1}}{66.1}
\]

When the unit load is at $L_4$, $R_4 = 0.667$

\[
P_{u4M_5} = \frac{1}{66.1} [1 \times 78 - 0.667 \times 54] = 0.635 \text{ (comp.)}
\]

When the unit load is at $L_5$, $R_4 = 0.583$

\[
P_{u4M_5} = \frac{1}{66.1} (0.583 \times 54) = 0.476 \text{ (tension)}
\]
The I.L. diagram for \( P_{U4M5} \) is shown in Fig. 4.10 (a).

(4) Influence line for \( P_{M5L6} \)

\( M_4L_5 \) is a secondary member, and hence will be stressed only when the unit load is in
the panel \( L_4L_6 \). The I.L. diagram will be a triangle having zero ordinates under \( L_4 \) and \( L_6 \),
and unit ordinate under \( L_5 \), as shown in Fig. 4.10 (e).

(5) Influence line for \( P_{M5U6} \)

\( M_4U_4 \) is also a secondary member, and will be stressed only when the unit load is in
the panel \( L_4L_6 \). Pass a horse shoe section \( bb \) cutting five members. Out of these, the lines
of action of four forces pass through \( L_6 \). Hence take moments about \( L_6 \) and consider the
equilibrium of portion enclosed by the horse shoe section.

Thus
\[ P_{M5U6} \times (15 \sin \phi) = M_{L6} \]
\[ \sin \phi = \frac{6}{\sqrt{(8.5)^2 + 6^2}} = \frac{6}{10.4} = 0.577 \]
\[ \cos \phi = \frac{8.5}{10.4} = 0.818 \]

\[ P_{M5U6} = \frac{M_{L6}}{15 \sin \phi} = \frac{M_{L6}}{15 \times 0.577} = \frac{M_{L6}}{8.65} \]

When the unit load is at \( L_4 \), \( M_{L4} = 0 \) and hence \( P_{M5U6} = 0 \).

When the unit load is at \( L_5 \), \( M_{L6} = 1 \times 6 = 6 \)

\[ \therefore P_{M5U6} = \frac{M_{L6}}{8.65} = \frac{6}{8.65} = 0.694 \text{ (tension)} \]

When the unit load is at \( L_6 \), \( M_{L6} = 0 \) and hence \( P_{M5U6} = 0 \).

The influence line diagram for \( P_{M5U6} \) is shown in Fig. 4.10 (f).

(6) Influence line for \( M_5L_4 \)

Pass a section \( cc \) cutting four members. Out of these, the line of action of forces in
two members pass through \( O_1 \). Hence consider the equilibrium of the portion to the left
of section \( cc \) and take moments about \( O_1 \).

(i) When the unit load is at \( L_4 \), \( R_A = 0.677 \) and \( P_{M5U6} = 0 \)

\[ (P_{M5L6} \sin \theta) \times O_1L_6 = 1 \times O_1L_4 - 0.667 \times O_1A \]

\[ P_{M5L4} = \frac{1}{0.734 \times 90} [1 \times 78 - 0.667 \times 54] = 0.636 \text{ (comp.)} \]

(ii) When the unit load is at \( L_5 \), \( R_A = 0.583 \) and \( P_{M5U6} = 0.694 \) (tension)

Taking moments about \( O_1 \), we get

\[ (P_{M5L6} \sin \theta) O_1L_6 = (1 \times O_1L_5) - (R_A \times O_1A) + P_{M5U6} \sin \phi \times 6.5) - (P_{M5U6} \cos \phi \times O_1L_5) \]

\[ P_{M5L6} = \frac{1}{0.734 \times 90} [(1 \times 84) - (0.583 \times 54) + (0.694 \times 0.577 \times 6.5) - (0.694 \times 0.818 \times 84)] \]

\[ = 0.113 \text{ (compression).} \]

When the unit load is at \( L_6 \), \( R_A = 0.5 \) and \( P_{M5U6} = 0 \)

\[ (P_{M5L6} \sin \theta) O_1L_6 = R_A \times O_1A \]

\[ \therefore P_{M5L6} = \frac{0.5 \times 54}{0.734 \times 90} = 0.408 \text{ (tension)} \]

The I.L. diagram for \( P_{M5L6} \) is shown in Fig. 4.10 (g).

(7) Influence line for \( P_{M3U4} \): The influence line for \( P_{M3U4} \) can be drawn exactly in the
same manner as that for \( P_{M5U6} \). The I.L. diagram is a triangle with a central ordinate of
0.8 as shown in Fig. 4.10 (h). Reader is advised to compute this ordinate.
(8) Influence line for \( P_{U4L4} \)
Pass a section \( dd \) cutting four members.
If \( \alpha \) is the inclination of member \( M_3U_4 \) with the vertical
\[
\sin \alpha = \frac{6}{\sqrt{8.5^2 + 6^2}} = \frac{6}{10.4} = 0.577 \quad \cos \alpha = \frac{8.5}{10.4} = 0.818
\]

Prolong \( U_4U_2 \) backwards to meet \( BA \) produced in \( O_2 \). Let \( \theta_2 \) be the inclination of \( U_4U_2 \) with horizontal.

Then \( \tan \theta_2 = \frac{13 - 9}{12} = \frac{4}{12} = \frac{1}{3} \); \( \theta_2 = 18^\circ 26' \)
\[
\sin \theta_2 = 0.316 \quad \text{and} \quad \cos \theta_2 = 0.949
\]
\[
O_2L_4 = \frac{U_4L_4}{\tan \theta_2} = 13 \times 3 = 39 \quad \text{m}
\]
\[
O_2A = 39 - 24 = 15 \quad \text{m}
\]
Take the moments about \( O_2 \) of all unbalanced forces to the left of the section \( dd \).

(i) When the unit load is at \( L_2 \), \( R_A = \frac{60}{72} = 0.833 \) and \( P_{M3U4} = 0 \)
\[
\therefore \quad P_{U4L4} = \frac{M_{O2}}{O_2L_4} = \frac{M_{O2}}{39} = \frac{1}{39} \{ (1 \times 27) - (0.833 \times 15) \} = 0.372 \quad \text{(tension)}.
\]

(ii) When the unit load is at \( L_3 \), \( R_A = 0.75 \) and \( P_{M3U4} = 0.8 \) (tension)
\[
\therefore \quad P_{U4L4} \times O_2L_4 = (P_{M3U4} \cos \alpha \times O_2L_4) = (P_{M3U4} \sin \alpha \times 13) + (R_A \times O_2A) - (1 \times O_2L_3)
\]
or
\[
P_{U4L4} = \frac{1}{39} (0.8 \times 0.818 \times 39) - (0.8 \times 0.577 \times 13) + (0.75 \times 15) - (1 \times 33)
\]
\[
= 0.058 \quad \text{(tension)}
\]

(iii) When the unit load is at \( L_4 \), \( R_A = 0.667 \) and \( P_{M3U4} = 0 \)
\[
\therefore \quad P_{U4L4} = \frac{1}{39} \{ (1 \times 39) - (0.067 \times 15) \} = 0.744 \quad \text{(tension)}.
\]

(iv) When the unit load is at \( L_5 \), \( R_A = 0.583 \) and \( P_{M3U4} = 0 \)
\[
\therefore \quad P_{U4L4} = \frac{1}{39} (0.583 \times 15) = 0.224 \quad \text{(comp.)}
\]

The influence line diagram for \( P_{U4L4} \) is shown in Fig. 4.10 (i).

4.11. PENNSYLVANIA OR PETTIT TRUSS WITH SUB-STRUTS

Fig. 4.11 (a) shows Pennsylvania or Pettit truss with sub-stra. The truss consists of 12 panels, each of 6 m length. Let us draw the influence lines of the members of the main panel \( L_4L_6 \).

1. Influence line for \( P_{L5L4} \) (and \( P_{L4L6} \))
Pass a section \( aa \) cutting three members \( U_4U_6 \), \( M_5L_6 \) and \( L_5L_6 \).
\[
\therefore \quad P_{L5L4} = \frac{M_{U4}}{U_4L_4} = \frac{M_{U4}}{13}
\]

When the unit load is at \( L_4 \), \( R_A = 8/12 \).
\[
\therefore \quad P_{L5L4} = \frac{1}{13} \left[ \frac{8}{12} \times 24 \right] = 1.231 \quad \text{(tension)}
\]
When the unit load is at \( L_5 \), \( R_A = \frac{7}{12} \)

\[
P_{L_5L_6} = \frac{1}{13} \left[ \frac{7}{12} \times 24 + 1 \times 6 \right] = 1.538 \text{ (tension)}
\]

**FIG. 4.11. PENNSYLVANIA OR PETTIT TRUSS WITH SUB-STRUTS.**
When the unit load is at \( L_6 \), \( R_4 = 1/2 \)

\[
P_{l5l6} = \frac{1}{13} \left[ \frac{1}{2} \times 12 \right] = 0.923 \text{ (tension)}
\]

When the unit load is either at \( A \) or at \( B \), \( M_{U_6} \) is zero and hence \( P_{l5l6} \) is zero. The I.L. for \( P_{l5l6} \) is shown in Fig. 4.11 (b). Since the stress in \( L_4 L_5 \) is equal to stress in \( L_5 L_6 \), this also represents the I.L. for \( P_{l4l5} \).

**2. Influence line for \( P_{u4u6} \):**

\[
P_{u4u6} = \frac{M_{L_6}}{x}
\]

where \( x \) is the perpendicular distance of \( U_4 U_6 \) from \( L_6 \). Prolong \( U_6 U_4 \) and \( L_6 L_5 \) to meet at \( O_1 \). Let \( \theta_1 \) be the inclination of \( U_4 U_6 \) with horizontal.

\[
\tan \theta_1 = \frac{15 - 13}{12} = 0.1667 \quad ; \quad \theta_1 = 9.462^\circ
\]

\[
\sin \theta_1 = 0.1644 \quad \text{and} \quad \cos \theta_1 = 0.9864
\]

Now

\[
O_1 L_6 = \frac{U_6 L_6}{\tan \theta_1} = \frac{15}{0.1667} = 89.98 \text{ m}
\]

Now

\[
x = O_1 A' = 89.98 - 36 = 53.98 \text{ m}
\]

\[
P_{u4u6} = \frac{M_{L_6}}{14.973}
\]

When the unit load is at \( L_6 \), \( R_4 = 1/2 \)

\[
P_{u4u6} = \frac{1}{14.973} \left[ \frac{1}{2} \times 36 \right] = 1.202 \text{ (compressive)}
\]

When the unit load is either at \( A \) or at \( B \), \( M_{L_6} \) is zero and hence \( P_{u4u6} \) is zero. The I.L. for \( P_{u4u6} \) will therefore be a triangle having zero ordinates at \( A \) and \( B \), and an ordinate of 1.202 under \( L_6 \), as shown in Fig. 4.11 (c).

**3. Influence line for \( P_{m5l6} \):**

\[
P_{m5l6} = \frac{M_{O_1}}{y}
\]

where \( y \) is the perpendicular distance of \( M_5 L_6 \) from \( O_1 \). Let \( \theta \) be the inclination of \( M_5 L_6 \) with \( O_1 L_6 \).

\[
\sin \theta = \frac{13}{\sqrt{13^2 + 12^2}} = 0.7348
\]

\[
y = O_1 L_6 \sin \theta = 89.98 \times 0.7348 = 66.12 \text{ m}
\]

\[
P_{m5l6} = \frac{M_{O_1}}{66.12}
\]

When the unit load is at \( L_4 \), \( R_4 = 8/12 \)

\[
P_{m5l6} = \frac{1}{66.12} \left[ 1 \times (89.98 - 12) - \frac{8}{12} \times 53.98 \right] = 0.635 \text{ (comp.)}
\]

When the unit load is at \( L_5 \), \( R_4 = 7/12 \)

\[
P_{m5l6} = \frac{1}{66.12} \left[ 1 \times (89.98 - 6) - \frac{7}{12} \times 53.98 \right] = 0.794 \text{ (comp.)}
\]

When the unit load is at \( L_6 \), \( R_4 = 1/2 \)

\[
P_{m5l6} = \frac{1}{66.12} \left[ \frac{1}{2} \times 53.98 \right] = 0.408 \text{ (tensile)}
\]

The complete I.L. for \( P_{m5l6} \) is shown in Fig. 4.11 (d), indicating that there is reversal of stress when the load traverses the span \( L_5 L_6 \).
4. Influence line for $P_{M_5 L_5}$

$M_5 L_5$ is a secondary member (sub-tie) and hence it carries stress only when the load is in panel $L_4 L_6$. When the unit load is at $L_5$, $P_{M_5 L_5}$ is evidently equal to unity (tensile). When the load is at $L_4$ or $L_6$, $P_{M_5 L_5}$ is evidently zero. The I.L. for $P_{M_5 L_5}$ will evidently be a triangle having zero ordinates at $L_4$ and $L_6$ and an ordinate of unity under $L_5$, as shown in Fig. 4.11 (c).

5. Influence line for $P_{L_4 M_5}$

$L_4 M_5$ is also a secondary member (sub-strut) and hence it carries stress only when the load is in panel $L_4 L_6$. Pass a horse shoe section $bb$, cutting five members $L_4 L_5$, $L_1 M_5$, $U_4 M_5$, $M_3 L_6$ and $L_5 L_6$. Out of these, lines of action of forces in four members (i.e. $L_4 L_5$, $U_4 M_5$, $M_3 L_6$ and $L_5 L_6$) pass through joint $L_6$. Consider the equilibrium of the portion of the truss within the horse-shoe section $bb$.

\[
P_{L_4 M_5} = \frac{M_{L_4}}{z}
\]

where $z$ is the perpendicular distance of $L_4 M_5$ from $L_6$.

\[
z = L_4L_6\sin\theta = 12 \times 0.7348 = 8.818 \text{ m}
\]

\[
P_{L_4 M_5} = \frac{M_{L_6}}{8.818}
\]

When the unit load is at $L_4$ (or at $L_6$) the load is out side the horse-shoe section. Hence $M_{L_6}$ and therefore $P_{L_4 M_5}$ is zero.

When the unit load is at $L_5$, $M_{L_6} = 1 \times 6 = 6$

\[
P_{L_4 M_5} = \frac{6}{8.818} = 0.68 \text{ (compression)}
\]

The I.L. for $P_{L_4 M_5}$ will therefore be a triangle having zero ordinates under $L_4$ and $L_6$ and an ordinate of 0.68 under $L_5$, as shown in Fig. 4.11 (f).

6. Influence line for $P_{U_4 M_5}$

Pass a section $cc$, cutting four members $U_4 U_6$, $U_4 M_5$, $L_4 M_5$ and $L_4 L_5$. Out of these, the lines of action of forces in $U_4 U_6$ and $L_4 L_5$ pass through $O_1$ while $P_{U_4 M_5}$ is known.

Now

\[
P_{U_4 M_5} = \frac{M_{O_1}}{y}
\]

where $y$ is the perpendicular distance of $U_4 M_5$ from $O_1 = 66.12$ m as found earlier.

When the unit load is at $L_4$, $R_A$ is 8/12 and $P_{L_4 M_5}$ is zero.

\[
P_{U_4 M_5} = \frac{1}{66.12} \left[ 1 \left(53.98 + 24\right) - \frac{8}{12} \times 53.98 \right] = 0.635 \text{ (compression)}
\]

When the unit load is at $L_5$, $R_A$ is 7/12 and $P_{L_4 M_5}$ is 0.68 (comp.)

\[
P_{U_4 M_5} = \frac{1}{66.12} \left[ 0.68 \times O_1 L_4 \sin\theta - \frac{7}{12} \times O_1 A \right]
\]

\[
= \frac{1}{66.12} \left[ 0.68 \left(53.98 + 24\right) \times 0.7348 - \frac{7}{12} \times 53.98 \right]
\]

\[= 0.113 \text{ (compression)}
\]

When the unit load is at $L_6$, $R_A = 6/12$ and $P_{L_4 M_5}$ is zero.

\[
P_{U_4 M_5} = \frac{1}{66.12} \left( \frac{6}{12} \times 53.98 \right) = 0.408 \text{ (tensile)}
\]

The I.L. for $P_{U_4 M_5}$ is shown in Fig. 4.12 (g).
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Taking moments about $A$, \[ R_B \times 32 - \left(1 - \frac{x}{48}\right) 56 = 0 \]
From which
\[ R_\theta = \frac{7}{4} \left( 1 - \frac{x}{48} \right) \uparrow \]
Hence
\[ R_A = W_C - R_\theta = \left( 1 - \frac{x}{48} \right) - \frac{7}{4} \left( 1 - \frac{x}{48} \right) \]
or
\[ R_A = 0.75 \left( \frac{x}{48} - 1 \right) \]
...\(a\)
When \(x = 0\), \(R_A = 0.75\) (↓) and \(R_\theta = 1.75\) (↑)
When \(x = 48\) m \(R_A = 0\) and \(R_\theta = 0\)
The I.L. diagram for \(R_A\) and \(R_\theta\) are shown in Fig. 4.12 (e) and (d) respectively.

2. Influence line for \(P_{U_3U_4}\): Pass a section \(aa\), cutting three members \(U_3U_4, L_3U_4\) and \(L_3L_4\) out of which members \(L_3L_4\) and \(L_3U_4\) meet at \(L_3\).
\[ P_{U_3U_4} = \frac{M_{L_3}}{U_3L_3} \]
Where
\[ U_3L_3 = 8 + \frac{16 - 8}{3} = 10.667 \text{ m} \]
Hence
\[ P_{U_3U_4} = M_{L_3}/10.667 \]
When the unit load is at \(U_3\), \(R_A = 1/2\) (↑)
\[ P_{U_3U_4} = \frac{1}{10.667} \left[ \frac{1}{2} \times 16 \right] = 0.75 \text{ (comp.)} \]
When the unit load is at \(B\), \(R_A = 0\). Hence \(M_{L_3}\), and therefore, \(P_{U_3U_4}\) is zero.
When the unit load is at \(C\), \(R_A = 0.75\) (↓)
\[ P_{U_3U_4} = \frac{1}{10.667} \left[ 0.75 \times 16 \right] = 1.125 \text{ (tension)} \]
When the load is at \(D\), \(R_A = 0\). Hence \(M_{L_3}\) and therefore, \(P_{U_3U_4}\) is zero. The I.L. diagram for \(P_{U_3U_4}\) is shown in Fig. 4.12 (e).

3. Influence line for \(P_{L_3L_4}\)
\[ P_{L_3L_4} = \frac{M_{U_4}}{y} \]
Where \(y\) is the perpendicular distance of \(L_3L_4\) from \(U_4\).
Now
\[ \tan \theta = \frac{16 - 8}{24} = \frac{1}{3} \Rightarrow \theta = 18.435^\circ \]
\[ \sin \theta = 0.3162 \text{ and } \cos \theta = 0.9487 \]
\[ OU_4 = \frac{U_2L_2}{\tan \theta} = \frac{8}{1/3} = 24 \text{ m} \]
\[ OA = 24 - 8 = 16 \text{ and } OU_4 = 16 + 24 = 40 \text{ m} \]
Now
\[ y = OU_4 \sin \theta = 40 \times 0.3162 = 12.648 \text{ m.} \]
Hence
\[ P_{L_3L_4} = \frac{M_{U_4}}{12.648} \]
When the unit load is at \(U_3\), \(R_A = \frac{1}{2}\) (↑)
\[ P_{L_3L_4} = \frac{1}{12.648} \left[ \frac{1}{2} \times 24 - 1 \times 8 \right] = 0.316 \text{ (tension)} \]
When the unit load is at \(U_4\), \(R_A = \frac{1}{4}\) (↑)
\[ P_{L_3L_4} = \frac{1}{12.648} \left[ \frac{1}{4} \times 24 \right] = 0.474 \text{ (tension)} \]
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You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
The first suffix to \( y \) denotes the point where the deflection is reckoned and the second suffix denotes the position of the unit point load. Thus, when the unit load is acting at \( X \), the deflection of point \( B \), in the absence of \( R_B \), will be equal to \( y_{XB} \). However, as the support at \( B \) is at the same level as \( A \) and \( C \), the upward deflection at \( B \) due to \( R_B \) is to neutralize this downward deflection \( y_{XB} \). Hence we get, from consistent deformation (chapter 7):

\[
R_B \frac{y_{XB}}{y_{AB}} = y_{BX}
\]

By Maxwell's reciprocal theorem (§7.3), \( y_{XB} = y_{XB} \)

Hence

\[
R_B = \frac{y_{XB}}{y_{AB}} \quad (5.1)
\]

Thus, the reaction at \( B \), due to unit load at any point \( X \) is proportional to the deflection at the point \( X \) due to the unit load acting at \( B \). In other words, the deflection curve shown in Fig. 5.1 (c) represents, to some scale, the influence line for \( R_B \).

If the deflection \( y_{AB} \) in the direction of unit load at \( B \), is selected as unity, the deflection curve will directly give influence line for \( R_B \).

### 5.3. Influence Lines for Statically Determinate Beams

The Müller-Breslau Principle is applicable both for statically determinate beams as well as for statically indeterminate beams. Let us first take statically determinate beams.

The Müller-Breslau influence theorem for statically determinate beams may be stated as follows:

"The influence line for an assigned function of a statically determinate beam may be obtained by removing the restraints offered by that function and introducing a directly related generalised unit displacement at the location and in the direction of the function."

Fig. 5.2 (a) shows a simply supported beam \( AB \) of span \( L \).

1. I.L. For reaction \( R_A \) and \( R_B \)

The I.L. for reaction \( (R_A) \) at \( A \) can be found by lifting

![Diagram of beam with influence lines](image)

**FIG. 5.2.**
the beam off the support $A$ by a unit distance, as shown in Fig. 5.2 (b). The deflected shape
gives the I.L. for $R_A$. This can be easily proved by applying the principle of virtual work to
the rigid body motion of the beam shown in Fig. 5.2 (b). The total virtual work ($\delta W$) must
be equal to zero since the resultant of the force system is zero. Thus, if the ordinate under
the unit load is $y$, we have
\[ \delta W = R_A (1.0) - 1.0(y) = 0 \]
which gives
\[ y = R_A, \] which proves the proposition.

When the unit load is at a distance $\alpha L$ from $A$, the magnitude of $y$ is given by the
relation
\[ \frac{1}{L} = \frac{y}{L - \alpha L} \]
\[ y = (1 - \alpha) \]

Similarly, the I.L. for reaction $R_B$ can be found, as shown in Fig. 5.2 (c).

2. I.L. for S.F. at $C$ : Let us find the I.L. for S.F. ($F_C$) at $C$. We know that S.F. ($F_C$) acts to both the
sides of the section and is represented by ( \( \downarrow \uparrow \) ). Hence cut the beam at
$C$ in two parts $AC$ and $CB$. The free body diagram of the two parts is shown in Fig. 5.3 (b). Let the beam
go through rigid body motions of parts $AC$ and $CB$, as shown in Fig.
5.2 (d), so that the total movement $C_1 C_2 = \text{unity}$. The deflected shape
will then give the influence line for $F_C$. This can be very easily proved
by applying the principle of virtual work. Thus, if $y$ is the ordinate of the I.L. under the unit load, we have
\[ \delta W = R_A (0) - 1.0(y) - M_C (\theta) - F_C (CC_1) + F_C (CC_2) + M_C (\theta) + R_B (0) = 0 \]
or
\[ y = F_C (CC_1 + CC_2) = F_C (C_1 C_2), \] where $C_1 C_2 = 1$
\[ y = F_C, \] which proves the proposition.

Now, if the section $C$ is at a distance $x$ from $A$, $CC_1$ will be equal to $x/L$. and $CC_2$ will
be equal to $(L - x)/L$. Similarly, the ordinate $y$ under the unit load is given by
\[ y = \left( \frac{x}{L} \right) \times \frac{\alpha L}{x} = \alpha \]

3. I.L. for B.M. ($M_C$) at $C$

For obtaining I.L. for $M_C$, introduce a hinge at $C$, and let the system go through rigid
body motions of $AC$ and $CB$ as shown in Fig. 5.2 (e). Then
\[ \delta W = R_A (0) - 1.0(y) - F_C (CC_1) + M_C (\theta_1) + F_C (CC_1) + M_C (\theta_2) + R_B (0) = 0 \]
or
\[ y = M_C (\theta_1 + \theta_2), \] where $\theta_1 + \theta_2 = 1$
\[ y = M_C \]
Now
\[ CC_1 = x \theta_1 = (L - x) \theta_2 \]
Thus,
\[ \theta_2 = \frac{x}{L - x} \theta_1 \]
But \[ \theta_1 + \theta_2 = 1 \]
\[ \therefore \theta_1 + \frac{x}{L - x} \theta_1 = 1 \quad \text{or} \quad \theta_1 \left( \frac{L}{L - x} \right) = 1 \quad \therefore \theta = \frac{L - x}{L} \]

Hence \[ CC_1 = \frac{x}{L} \theta_1 = \frac{x}{L} (L - x) \]

Also, ordinate \( y \) is given by \[ \frac{y}{\alpha L} = \frac{CC_1}{x} \]
or
\[ y = \alpha L \times \frac{1}{x} \times \frac{x}{L} (L - x) = \alpha (L - x) \]

When the unit load is at \( C \), \( \alpha L = x \), or \( \alpha = x/L \)
\[ y = CC_1 = \frac{x}{L} (L - x) \]

Example 5.1. A two span beam \( ABC \) has internal hinges at \( D \) and \( E \). Using Müller-Breslau influence theorem, sketch I.L. for (i) \( R_A \) (ii) \( R_B \) (iii) \( R_C \) and (iv) \( M_C \).

**Solution.** If there were no hinges at \( D \) and \( E \), the beam would be statically indeterminate to second degree. However, provision of hinges at \( D \) and \( E \) makes the beam statically determinate.

(i) I.L. For \( R_A \): According to Müller-Breslau Theorem, in order to find I.L. for \( R_A \), lift the beam off the support \( A \) by unity in the direction of \( R_A \). The deflected shape of the beam, shown in Fig. 5.4 (b), gives the I.L. for \( R_A \). It should be noted that because of a hinge at \( D \), only the portion \( AD \) will be lifted up, and the remaining portion will remain horizontal. This suggests that when the unit load crosses \( D \), \( R_A \) will be zero and will continue to remain zero as the unit load moves along \( DBEC \).

(ii) I.L. For \( R_B \): For \( R_B \), lift the beam off the support \( B \) by unity. The beam will deflect as shown in Fig. 5.4 (c), which will be I.L. for \( R_B \). Ordinate \( DD_1 \) will evidently be equal to \( \frac{1}{L/2} \times L = 2 \). When the unit load crosses \( E \), \( R_B \) will be zero.

(iii) I.L. For \( R_C \). Lift the beam off the support \( C \) by unity. The beam

**FIG. 5.4.**
will deflect as shown in Fig. 5.4 (d) which will be the I.L. for $R_C$, according to the Müller-Breslau principle. Since $CC_1 = 1$, $E$ will move to $E_1$ such that $EE_1 = 1$.

Hence, from geometry, $DD_1 = 1$ in negative direction. This suggests that when the unit load is between $A$ to $B$, the reaction $R_C$ will be negative i.e. $R_C$ will act downwards.

(iv) I.L. for $M_C$. Let us assume $M_C$ to be in clockwise direction. Hence in order to find I.L. for $M_C$, introduce a hinge at $C$ and rotate the beam, at $C$ by $\theta = 1$ unit. The beam will deflect as shown in Fig. 5.4 (e) which will evidently be the I.L. for $M_C$.

\[
\text{Ordinate } EE_1 = \frac{EC}{\theta} = \frac{L/2}{1} = \frac{L}{2}.
\]

Hence by geometry, $DD_1 = \frac{L}{2}$.

When the unit load is between $A$ to $B$, $M_C$ will be negative, i.e. it will act in the counterclockwise direction. For the unit load positions between $B$ and $C$, $M_C$ will act in the clockwise direction, as marked in Fig. 5.4 (a).


Solution.

(i) I.L. for $R_A$. For getting I.L. for $R_A$, lift the beam off the support $A$ by unity. Due to internal hinge at $B$, only portion $AB$ will be deflected, as shown in Fig. 5.5 (b) which is the I.L. for $R_A$. The reaction $R_A$ remains zero for load positions between $B$ to $E$.

(ii) I.L. for $R_C$. Lift the beam off the support $C$ by unity. The deflected shape, shown in Fig. 5.5(c), will be the I.L. for $R_C$, as per Müller-Breslau principle. The ordinate $BB_1$, is given by

\[
BB_1 = \frac{CC_1}{CD} \times BD = \frac{1}{L} \times \frac{3}{2} L = 1.5 \text{ (positive)}.
\]

Similarly

\[
EE_1 = CC_1 = 1 \text{ (negative)}.
\]

FIG. 5.5
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You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
Let the unit load roll on the span $ABC$. At any instant, let it be at a point $X$. We want to plot the influence line for bending moment $M_D$ at the point $D$. According to Müller-Breslau principle the internal stress component, for which the influence line is to be plotted, is first removed. For the present case, this is accomplished by inserting a pin at $D$. The beam will then deflect, under the unit load at $X$, as shown in Fig. 5.10 (b). Let $\phi'_{DX}$ be the rotation at $D$, due to unit load at $X$. Now remove the unit load from $X$, and apply a pair of unit moments at $D$, as shown in Fig. 5.10 (c).

Let $\phi_{DD} = \text{rotation at } D$, due to unit couple at $D$.

$y'_{XD} = \text{deflection at } X$ due to unit couple at $D$.

From method of consistent deformation (Chapter 7), we have $M_D \cdot \phi_{DD} = \phi'_{DX}$ where $M_D = \text{bending moment at } D$ due to unit load at $X$.

\[ M_D = \frac{\phi'_{DX}}{\phi_{DD}} \]

But from reciprocal theorem, $\phi'_{DX} = y'_{XD}$ (see Eq. 7.7)

\[ M_D = \frac{y'_{XD}}{\phi_{DD}} \quad \ldots (5.6) \]

Eq. 5.6 suggests that $M_D$ is proportional to $y'_{XD}$. In other words, the deflection curve of Fig. 5.10 (c) gives, to some scale, the influence line for $M_D$. If however, $\phi_{DD}$ is selected to be unity, the deflection at any point $X$ will give the bending moment at $D$. Eq. 5.6, therefore, further proves the validity of the Müller-Breslau principle applied to the influence line for bending moment.

5.6. CONTINUOUS BEAM: INFLUENCE LINE FOR SHEAR FORCE

Let us now study the applicability of the Müller-Breslau principle for plotting the influence line for the shear force at any point $D$ of a continuous beam $ABC$ shown in Fig. 5.11. Let the unit load be at any point $X$. In order to remove the internal stress components, i.e. shear $F_D$ at $D$, assume that the beam is cut at $D$ and that a slide device inserted in such a way that it permits relative transverse deflection between the two parts of the cut, as shown, in Fig. 5.11 (b), but which at the same time, maintains a common slope at both the ends of the cut.
Now remove the unit load from $X$, and apply a pair of unit loads (shear) at $D$. The beam will deflect as shown in 5.11 (c).

Let $F_D = \text{shear force at } D$.

$Y_{DX} = \text{relative linear deflection at } D \text{ due to unit load at } X$.

$Y_{DD} = \text{relative linear deflection at } D \text{ due to pair of unit load (shear) at } D$.

$Y_{XD} = \text{deflection at } X \text{ due to pair of unit load (shear) at } D$.

Then, from compatibility of deformation at $D$, we have

$$F_D \cdot Y_{DD} = Y_{DX} \text{ or } F_D = \frac{Y_{DX}}{Y_{DD}}$$

But from the reciprocal theorem,

$$Y_{DX} = Y_{XD}$$

$$\therefore F_D = \frac{Y_{XD}}{Y_{DD}} \quad \ldots(5.7)$$

Thus, $F_D$ is proportional to $Y_{XD}$. In other words, the deflection curve of Fig. 5.11 (c) gives the influence line for shear at $D$, to some scale. If, however, $Y_{DD}$ is taken as unity, the deflection at any point $X$ of Fig. 5.11 (c) gives that the shear at $D$ due to unit vertical load at $X$.

5.7. INFLUENCE LINE FOR HORIZONTAL REACTION

Let us now study the influence line for horizontal reaction at the hinged end $A$ of a frame shown in Fig. 5.12 (a). The unit vertical load can travel on $BC$, or unit horizontal load can travel on $AB$. Let us first take the case when the unit vertical load travel on $BC$.

According to the Müller-Breslau principle, the reaction component at $A$ is first removed. This is accomplished by supporting end $A$ on rollers. The end $A$ will deflect horizontally by $\Delta_{AA}$ due to unit vertical load at $X$, as shown in Fig. 5.12 (b). The unit vertical load at $X$ is then removed and a unit horizontal load is applied at $A$. The frame will deflect as shown in Fig. 5.12 (c). Let

$\Delta_{AA} = \text{Horizontal deflection of } A \text{ due to horizontal unit load at } A$.

$\Delta_{XA} = \text{Vertical deflection at } X \text{ due to unit horizontal load at } A$.

$H_A = \text{Horizontal reaction at } A$.

Then $H_A \Delta_{AA} = \Delta_{AX}$

or $H_A = \frac{\Delta_{AX}}{\Delta_{AA}}$
But \[ \Delta_{AX} = \Delta_{XA} \] from reciprocal theorem

Hence \[ H_A = \frac{\Delta_{XA}}{\Delta_{AA}} \] \hspace{1cm} \ldots(5.8)

Eq. 5.8 shows that the deflection curve of $BC$ [Fig. 5.12 (c)] gives the influence line, to some scale for $H_A$ when a unit vertical load moves on $BC$. Similarly, it can be shown that the deflection curve of $AB$ gives the influence line for $H_A$ when a unit horizontal load moves on $AB$.

**Example 5.3.** Draw the influence lines for (i) reaction at $B$; and (ii) moment at $A$ for the propped cantilever shown in Fig. 5.13 (a). Compute the ordinates at intervals of 1.25 m.

**Solution**

(a) **I.L. for $R_B$**

From Eq. 5.2, \[ R_B = \frac{y_{XB}}{y_{BB}} \] \hspace{1cm} \ldots(1)

To compute $y_{XB}$, apply a unit vertical load at $B$, as shown in Fig. 5.13 (b).

At any section $X$ distant $x$ from $B$, we have

\[ EI \frac{dy}{dx^2} = -M_x = 1.0 \cdot x \]

Integrating,

\[ EI \frac{dy}{dx} = \frac{x^2}{2} + C_1 \]

At \[ x = L, \frac{dy}{dx} = 0 \]
\[ C_1 = - \frac{L^2}{2} \]

Hence \[ EI \frac{dy}{dx} = \frac{x^2}{2} - \frac{L^2}{2} \]

Integrating further,

\[ EIy = \frac{x^3}{6} - \frac{L^2}{2} x + C_2 \]

At \[ x = L, y = 0 \]
\[ C_2 = \frac{L^3}{2} - \frac{L^3}{6} = \frac{L^3}{3} \]

Hence \[ EIy = \frac{x^3}{6} - \frac{L^2}{2} x + \frac{L^3}{3} \] \hspace{1cm} \ldots(2)

At \[ x = 0, y = y_{BB} = \frac{L^3}{3EI} \]

At \[ x = x, \]

\[ y_{XB} = \frac{1}{EI} \left( \frac{x^3}{6} - \frac{L^2}{2} x + \frac{L^3}{3} \right) \]

Substituting these in (1), we get

\[ R_B = \left( \frac{x^3}{6} - \frac{L^2}{2} x + \frac{L^3}{3} \right) \frac{3}{L^3} \]

**FIG. 5.13.**
\[ R_B = \frac{1}{2} \left( \frac{x^3}{L^3} - \frac{3x}{L} + 2 \right) \]

or
\[ R_B = \frac{1}{2} \left( n^3 - 3n + 2 \right), \text{ where } \frac{x}{L} = n. \]

The ordinates of I.L. for \( R_B \) are computed in Table 5.1.

<table>
<thead>
<tr>
<th>( x , (m) )</th>
<th>0</th>
<th>1.25</th>
<th>2.50</th>
<th>3.75</th>
<th>5</th>
<th>6.25</th>
<th>7.5</th>
<th>8.75</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = \frac{x}{L} )</td>
<td>0</td>
<td>0.125</td>
<td>0.25</td>
<td>0.375</td>
<td>0.5</td>
<td>0.625</td>
<td>0.75</td>
<td>0.875</td>
<td>1</td>
</tr>
<tr>
<td>( R_B )</td>
<td>1</td>
<td>0.813</td>
<td>0.633</td>
<td>0.464</td>
<td>0.313</td>
<td>0.185</td>
<td>0.087</td>
<td>0.023</td>
<td>0</td>
</tr>
</tbody>
</table>

The I.L for \( R_B \) is shown in Fig. 5.9 (c).

(b) I.L. for \( M_A \)

In order to draw the I.L for \( M_A \), replace the fixed support at \( A \) by a pin, as shown in Fig. 5.13 (d). Remove the external unit load and apply a unit couple at \( A \), as shown in Fig. 5.13 (e). Then from Eq. 5.3.

\[ M_A = \frac{y'x_A}{\phi_{AA}} \quad \ldots(3) \]

where \( y'x_A \) = vertical deflection at \( X \) due to unit couple at \( A \)

\( \phi_{AA} \) = slope at \( A \) due to unit couple at \( A \),

Let \( R'_B = \) Reaction at \( B \), when unit moment is acting at \( A = \frac{1}{L} \uparrow \)

\[ \therefore \quad E I \frac{d^2y}{dx^2} = -M_x = -R'_B \cdot x = -\frac{x}{L} \]

\[ E I \frac{dy}{dx} = -\frac{x^2}{2L} + C_1 \]

and

\[ E I y = -\frac{x^3}{6L} + C_1 x + C_2 \]

At \( x = 0, y = 0 \) \quad \therefore \quad C_2 = 0 \quad ; \quad \text{At } x = L, y = 0 \quad \therefore \quad C_1 = L/6 \]

Hence

\[ E I \frac{dy}{dx} = -\frac{x^2}{2L} + \frac{L}{6} \quad \ldots(i) \]

and

\[ E I y = -\frac{x^3}{6L} + \frac{Lx}{6} \quad \ldots(ii) \]

At \( x = L, \frac{dy}{dx} = \phi_{AA} = \frac{1}{E I} \left( -\frac{L^2}{2L} + \frac{L}{6} \right) = -\frac{L}{3} \quad \ldots(4) \]

At \( x = x, y = y'x_A = \frac{1}{E I} \left( -\frac{x^3}{6L} + \frac{Lx}{6} \right) \quad \ldots(5) \]

Substituting these values in (3), we get

\[ M_A = \left( \frac{x^3}{6L} - \frac{Lx}{6} \right) \times \frac{3}{L} = \frac{1}{2} \left( \frac{x^2}{L^2} - x \right) \]

This is thus the equation of the influence line for \( M_A \). The ordinates are calculated in the tabular form in Table 5.2.
TABLE 5.2

<table>
<thead>
<tr>
<th>x (m)</th>
<th>0</th>
<th>1.25</th>
<th>2.5</th>
<th>3.75</th>
<th>5</th>
<th>6.25</th>
<th>7.5</th>
<th>8.75</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 / L^2 )</td>
<td>0</td>
<td>0.0195</td>
<td>0.156</td>
<td>0.53</td>
<td>1.25</td>
<td>2.45</td>
<td>4.24</td>
<td>6.7</td>
<td>10.0</td>
</tr>
<tr>
<td>( M_A )</td>
<td>0</td>
<td>-0.615</td>
<td>-1.172</td>
<td>-1.61</td>
<td>-1.875</td>
<td>-1.9</td>
<td>-1.63</td>
<td>-1.025</td>
<td>0</td>
</tr>
</tbody>
</table>

The minus sign shows that the direction of \( M_A \) is in reverse direction to that of the unit moment applied at \( A \), i.e., \( M_A \) acts in anti-clockwise direction. The I.L. for \( M_A \) is shown in Fig. 5.13 (f).

**Example 5.4.** Determine the influence line for \( R_A \) for the continuous beam shown in Fig. 5.14. Compute the ordinates at every 1 m interval.

**Solution**

Apply a unit vertical load at \( A \), as shown in Fig. 5.14 (c). Then

\[
R_A = \frac{Y_{XA}}{y_{AA}}
\]

From Fig. 5.10 (c),

\[
R_B = 2 \quad \text{and} \quad R_C = 1
\]

\[
EI \frac{d^2y}{dx^2} = -M_x = -1 \times x \quad \mid + R_B (x-4)
\]

\[
= -x \quad \mid + 2 (x-4)
\]

\[
EI \frac{dy}{dx} = -\frac{x^2}{2} + C_1 \quad \mid + (x-4)^2
\]

and

\[
Ely = -\frac{x^3}{6} + C_1 x + C_2 \quad \mid + \frac{(x-4)^3}{3}
\]

At \( x = 4 \) m, \( y = 0 \),

\[
\therefore \ 4C_1 + C_2 = \frac{64}{6}
\]

At \( x = 8 \) m, \( y = 0 \),

\[
\therefore \ 8C_1 + C_2 = 64
\]

\[
\therefore \ C_1 = \frac{40}{3} \quad \text{and} \quad C_2 = -\frac{128}{3}
\]

Hence

\[
Ely = -\frac{x^3}{6} + \frac{40}{3} x - \frac{128}{3} \quad \mid + \frac{(x-4)^3}{3}
\]

At \( x = 0 \), \( y = y_{AA} = \frac{1}{EI} \left( -\frac{128}{3} \right) = -\frac{128}{3Ei} \)

At \( x = x \), \( y = y_{XA} = \left[ -\frac{x^3}{6} + \frac{40}{3} x - \frac{128}{3} \right] + \frac{(x-4)^3}{3} \left( \frac{1}{EI} \right)
\]

The values of \( y_{XA} \), for various of \( x \) are given in Table 5.3. The term \( \frac{1}{EI} \) has been omitted for convenience.
You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
order to find \( y'_{XD} \) (i.e. to determine the deflection curve), we shall use the conjugate beam method. Fig. 5.15 (d) shows a corresponding conjugate beam loaded with \( +\, M/EI \) diagram. Thus, since \( M_{\text{max}} = 2 \, \text{kN-m} \), the loading on the conjugate beam will be triangular, having a maximum intensity of \( +2 \) (i.e., acting downwards) at \( B \), \( EI \) being omitted. Since the real beam [Fig. 5.15 (c)] has a zero deflection at \( B \), the conjugate beam will have a pin at the corresponding point \( B' \) so that the B.M. there, representing deflection at \( B \), is zero. Let us first determine the reaction \( R_A' \), \( R_D' \) and \( R_C' \) of the conjugate beam. Taking moments about the pin \( B' \) and considering the equilibrium of the left portion, we get

\[
R_A' = \frac{1}{4} \left( \frac{1}{2} \times 4 \times 2 \times \frac{4}{3} \right) = \frac{4}{3} \uparrow
\]

Considering the equilibrium of the right portion, we have

\[
2R_D' + 4R_C' = \frac{1}{2} \times 4 \times 2 \times \frac{4}{3}
\]

or

\[
R_D' + 2R_C' = \frac{8}{3}
\]

Also,

\[
R_A' + R_D' + R_C' = \frac{1}{2} \times 8 \times 2 = 8
\]

or

\[
R_D' + R_C' = 8 - R_A' = 8 - \frac{4}{3} = \frac{20}{3}
\]

From (i) and (ii), we get \( R_D' = \frac{32}{3} \uparrow \) and \( R_C' = -4 \), i.e. acting (\( \downarrow \))

Now \( y'_{XD} \) of real beam = \( M_X \) of conjugate beam

and

\[
\phi_{DD} = \text{relative change in angle at } D
\]

\[
= \text{sum of shears in conjugate beam to the right and left of the support } D = R_D' = \frac{32}{3}.
\]

The calculation of \( M_X \) (and hence \( y'_{XD} \)) and \( M_D \) are done in the tabular form below. (Table 5.4). The influence line diagram for \( M_D \) is shown plotted in Fig. 5.15 (e).

**Alternative Solution**. The deflection curve or the value of \( y'_{XD} \) can also be determined by the conventional double integration method used in example 5.3.

Refer Fig. 5.15 (c), where \( R_A = \frac{1}{2} \uparrow \); \( R_B = 1 \downarrow \) and \( R_C = \frac{1}{2} \uparrow \). Since there is discontinuity in the beam at the pin at \( D \), we will treat the spans \( AD \) and \( DC \) separately.
TABLE 5.4.

<table>
<thead>
<tr>
<th>x (m)</th>
<th>( M_x = y' x D )</th>
<th>( M_D = \frac{y' x D}{\theta_{x D}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((-\frac{4}{3} \times 1) + (\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{3}) = -\frac{15}{12})</td>
<td>(-\frac{15}{12} \times \frac{3}{32} = -0.177)</td>
</tr>
<tr>
<td>2</td>
<td>((-\frac{4}{3} \times 2) + (\frac{1}{2} \times 2 \times 1 \times \frac{2}{3}) = -2)</td>
<td>(-2 \times \frac{3}{32} = -0.1875)</td>
</tr>
<tr>
<td>3</td>
<td>((-\frac{4}{3} \times 3) + (\frac{1}{2} \times 3 \times \frac{3}{2} \times \frac{3}{3}) = -\frac{7}{4})</td>
<td>(-\frac{7}{4} \times \frac{3}{32} = -0.164)</td>
</tr>
<tr>
<td>4</td>
<td>((-\frac{4}{3} \times 4) + (\frac{1}{2} \times 4 \times 2 \times \frac{4}{3}) = 0)</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>((4 \times 3) - (\frac{32}{3} \times 1) + (\frac{1}{2} \times 3 \times \frac{3}{2} \times \frac{3}{3}) = \frac{43}{12})</td>
<td>(\frac{43}{12} \times \frac{3}{32} = 0.336)</td>
</tr>
<tr>
<td>6</td>
<td>((4 \times 2) + (\frac{1}{2} \times 2 \times 1 \times \frac{2}{3}) = \frac{26}{3})</td>
<td>(\frac{26}{3} \times \frac{3}{32} = 0.813)</td>
</tr>
<tr>
<td>7</td>
<td>((4 \times 1) + (\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{3}) = \frac{49}{12})</td>
<td>(\frac{49}{12} \times \frac{3}{32} = 0.383)</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(i) For the span AD. Measuring x from the L.H. support A, we get

\[
EI \frac{d^2 y}{dx^2} = -M_x = -\frac{x}{2} \left| + (x - 4) \right.
\]

\[
\therefore \quad EI \frac{dy}{dx} = -\frac{x^2}{4} + C_1 \left| + \frac{(x - 4)^2}{2} \right.
\]

and

\[
Ely = -\frac{x^3}{12} + C_1 x + C_2 \left| + \frac{(x - 4)^3}{6} \right.
\]

At \( x = 0 \), \( y = 0 \) \quad C_2 = 0 ;

At \( x = 4 \) m, \( y = 0 \) = \(-\frac{64}{12} + 4C_1 \)

\( \therefore \quad C_1 = \frac{64}{12 \times 4} = \frac{4}{3} \)

Hence the slope and deflection equations for span AD are:

\[
EI \frac{dy}{dx} = -\frac{x^2}{4} + \frac{4}{3} \left| + \frac{(x - 4)^2}{2} \right. \quad \ldots \text{(I)}
\]

and

\[
Ely = -\frac{x^3}{12} + \frac{4}{3} x \left| + \frac{(x - 4)^3}{3} \right. \quad \ldots \text{(II)}
\]

At \( x = 6 \) m,

\[
Ely = -\frac{216}{12} + \frac{24}{3} + \frac{8}{6} = -\frac{26}{3}
\]

and

\[
(EI \frac{dy}{dx})_{DA} = -9 + \frac{4}{3} + 2 = -\frac{17}{3}
\]

(ii) For the span DC. Shear at pin D, just to its right, is equal to \( \frac{1}{2} \downarrow \) (i.e. shear at D is equal and opposite to \( R_c \)). Measuring x from D, to right

\[
EI \frac{d^2 y}{dx^2} = -M_x = \frac{1}{2} \cdot x - 1
\]
\[
EI \frac{dy}{dx} = \frac{x^2}{4} - x + C_1
\]
and
\[
EI y = \frac{x^3}{12} - \frac{x^2}{2} + C_1 x + C_2
\]

At \( x = 0 \), \( EI y = -\frac{26}{3} \) \(:\) \( C_2 = -\frac{26}{3} \).

At \( x = 2 \) m, \( EI y = 0 = \frac{8}{12} - \frac{4}{2} + 2C_1 - \frac{26}{3} \) \(:\) \( C_1 = +5 \).

Hence the slope and deflection equation for span \( DC \) are:

\[
EI \frac{dy}{dx} = \frac{x^2}{4} - x + 5
\]

and

\[
EI y = \frac{x^3}{12} - \frac{x^2}{2} + 5x - \frac{26}{3}
\]

At \( x = 0 \), \( \left( EI \frac{dy}{dx}\right)_{DC} = +5 \)

Now, \( \phi_{DD} = \text{relative change in this angle at } D \)

\[
= \left( \frac{dy}{dx} \right)_{DA} - \left( \frac{dy}{dx} \right)_{DC} = \frac{1}{EI} \left[ -\frac{17}{3} - 5 \right] = -\frac{1}{EI} \frac{32}{3}.
\]

The calculation of \( y'_{XD} \) is done in table 5.5. The term \( \frac{1}{EI} \) has been omitted for convenience.

<table>
<thead>
<tr>
<th>Distance from A (m)</th>
<th>Eq. No.</th>
<th>( y'_{XD} )</th>
<th>( M_D = \frac{y'<em>{XD}}{\phi</em>{DD}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( (x = 0) )</td>
<td>II</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1 ( (x = 1) )</td>
<td>II</td>
<td>-4/3 + 1/12</td>
<td>-15/12 \times 3/32 = -0.117</td>
</tr>
<tr>
<td>2 ( (x = 2) )</td>
<td>II</td>
<td>-8/12 + 8/3</td>
<td>-2 \times 3/32 = -0.1875</td>
</tr>
<tr>
<td>3 ( (x = 3) )</td>
<td>II</td>
<td>-27/12 + 12/3 + 7/4</td>
<td>-7/4 \times 3/32 = -0.164</td>
</tr>
<tr>
<td>4 ( (x = 4) )</td>
<td>II</td>
<td>-64/12 + 12/3</td>
<td>0</td>
</tr>
<tr>
<td>5 ( (x = 5) )</td>
<td>II</td>
<td>-125/12 + 20/3 + 6/1 = -43/12</td>
<td>43/12 \times 3/32 = 0.336</td>
</tr>
<tr>
<td>6 ( (x = 6) )</td>
<td>IV</td>
<td>-216/12 + 24/3 + 8/6 = -26/3</td>
<td>26/3 \times 3/32 = 0.813</td>
</tr>
<tr>
<td>7 ( (x = 1) )</td>
<td>IV</td>
<td>1/12 + 1/2 + 5 - 26/3 = -49/12</td>
<td>49/12 \times 3/32 = 0.383</td>
</tr>
<tr>
<td>8 ( (x = 2) )</td>
<td>IV</td>
<td>8/12 - 4/2 + 10 - 26/3 = 0</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 5.6. Determine the influence line for the shear force at D, the middle point of span BC, of a continuous beam shown in Fig. 5.16 (a). Compute the ordinates at 1 m interval.

Solution

In order to plot the influence line for shear force $F_D$, the shearing resistance of the beam is first removed by inserting a sliding device which permits the relative movement between the two parts but does not impair the moment resistance. Thus the slide device is such that it maintains the same slope in the distorted beam to either side of the device. The unit load (external) is removed and a pair of the unit loads (unit shear) is applied at $D$. The beam will then distort as shown in Fig. 5.16 (c). The S.F. at $D$ is given by

$$F_D = \frac{Y_{XD}}{Y_{DD}} \quad \ldots (1)$$

where $Y_{XD}$ = deflection of beam at $X$ due to unit shear at $D$.

$Y_{DD}$ = total relative movement at $D$ due to unit shear at $D$.

Considering the equilibrium of the portion $DC$ to the right of the cut, we have

$$R_C = 1 \uparrow \quad \text{and}$$

$$M_D = 1 \times 2 = 2 \quad \text{kN-m} \quad \ldots$$

Similarly, considering the equilibrium of the portion to the left of the cut, and taking moments about $A$, we get

$$4R_B (\downarrow) = M_D + 1 \times 6 = 2 + 6 = 8 \quad \therefore \quad R_B = 2 \downarrow$$

and

$$R_A = 2 - 1 = 1 \uparrow$$

The bending moment diagram will be a triangle having a maximum ordinate of 4 kN-m at $B$.

(a) Solution by conjugate beam method

Fig. 5.16(d) shows the conjugate beam loaded with $M/EI$ diagram. Omitting $EI$ for convenience, the loading diagram will also be triangle having maximum ordinate of 4 at $B$, the load acting downwards. In addition to this loading, an unknown moment load $\mu$ will also act at $D'$ of the conjugate beam, to satisfy the condition that the slope at both the sides of the slide real beam is the same. Since the slope at the real beam is represented by the shear of the conjugate beam, the shear just to the right of $D'$ must be equal to the shear to the left of $D'$. This
condition is satisfied by the moment \( \mu \) acting at \( D' \) of the conjugate beam. There is a pin \( B' \) corresponding to the support \( B \) of the real beam.

Consider the equilibrium of the portion to the left of hinge \( B' \).

\[
R_\alpha' = \frac{1}{4} \left( \frac{1}{2} \times 4 \times 4 \times \frac{4}{3} \right) = \frac{8}{3} 
\]
\[
R_c' = \text{total load} - R_\alpha' = \left( \frac{1}{2} \times 8 \times 4 \right) - \frac{8}{3} = \frac{40}{3}
\]

Again, considering the equilibrium of the portion to the right of pin \( B' \), and taking moment about \( B' \), we get

\[
\mu + \left( \frac{1}{2} \times 4 \times 4 \times \frac{4}{3} \right) = \frac{40}{3} \times 4
\]

or

\[
\mu = \frac{160}{3} - \frac{32}{3} = \frac{128}{3}
\]

Since the bending moment of the conjugate beam represent the deflection of the corresponding point of the real beam, the moment \( \mu \) represents the relative movement of the two parts of the slide. Hence

\[ Y_{DD} = \mu = \frac{128}{3} \]

The calculations of \( M_X \) (and hence \( Y_{XO} \)) and \( F_D \) are done in the tabular form below (Table 5.6).

<table>
<thead>
<tr>
<th>( x ) (m)</th>
<th>( M_X = Y_{XO} )</th>
<th>( F_D = \frac{Y_{XO}}{Y_{DD}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \left( - \frac{8}{3} \times 1 \right) + \left( \frac{1}{2} \times 1 \times 1 \times \frac{1}{3} \right) = -\frac{5}{2} )</td>
<td>( -\frac{5}{2} \times \frac{3}{2} \times 128 = -0.059 )</td>
</tr>
<tr>
<td>2</td>
<td>( \left( - \frac{8}{3} \times 2 \right) + \left( \frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \right) = -4 )</td>
<td>( -4 \times \frac{3}{128} = -0.094 )</td>
</tr>
<tr>
<td>3</td>
<td>( \left( - \frac{8}{3} \times 3 \right) + \left( \frac{1}{2} \times 3 \times 3 \times \frac{3}{3} \right) = -3.5 )</td>
<td>( -3.5 \times \frac{3}{128} = -0.082 )</td>
</tr>
<tr>
<td>4</td>
<td>( \left( - \frac{8}{3} \times 4 \right) + \left( \frac{1}{2} \times 4 \times 4 \times \frac{4}{3} \right) = -0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>5</td>
<td>( \left( - \frac{40}{3} \times 3 \right) + \left( \frac{1}{2} \times 3 \times 3 \times \frac{3}{3} \right) + \frac{128}{3} = +\frac{52}{3} )</td>
<td>( \frac{43}{6} \times \frac{3}{128} = +0.168 )</td>
</tr>
<tr>
<td>6 (left)</td>
<td>( \left( - \frac{40}{3} \times 2 \right) + \left( \frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \right) + \frac{128}{3} = +\frac{52}{3} )</td>
<td>( \frac{52}{3} \times \frac{3}{128} = +0.406 )</td>
</tr>
<tr>
<td>6 (right)</td>
<td>( \left( - \frac{40}{3} \times 2 \right) + \left( \frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \right) = -\frac{76}{3} )</td>
<td>( -\frac{76}{3} \times \frac{3}{128} = -0.594 )</td>
</tr>
<tr>
<td>7</td>
<td>( \left( - \frac{40}{3} \times 1 \right) + \left( \frac{1}{2} \times 1 \times 1 \times \frac{1}{3} \right) = -\frac{79}{6} )</td>
<td>( -\frac{79}{6} \times \frac{3}{120} = -0.308 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The I.L. for \( F_D \) is shown in Fig. 5.16(d).
(b) Alternative Solution

We shall now determine the values of $Y_{XD}$ for various values of $x$, by the double integration method. Since the beam is discontinuous at $D$, we will write the differential equations for both the portions separately. Refer Fig. 5.16(c). The reactions, calculated earlier, are as follows:

$$R_A = 1 \uparrow \quad R_B = 2 \downarrow \quad \text{and} \quad R_C = 1 \uparrow$$

**(i) For portion AD**

Measuring $x$ from $A$, towards right,

$$EI \frac{d^2 y}{dx^2} = -M_x = -x \bigg| + 2(x - 4)$$

$$\therefore \quad EI \frac{dy}{dx} = -\frac{x^2}{2} + C_1 \bigg| + (x - 4)^2$$

and

$$EI y = -\frac{x^3}{6} + C_1 x + C_2 \bigg| + \frac{(x - 4)^3}{3}$$

At $x = 0, \quad y = 0 \quad \therefore \quad C_2 = 0$

At $x = 4, \quad y = 0 = -\frac{64}{6} + 4C_1 \quad \therefore \quad C_1 = \frac{64}{24} = \frac{8}{3}$

Hence the slope and deflection equations for portion $AD$ are

$$EI \frac{dy}{dx} = -\frac{x^2}{2} + \frac{8}{3} \bigg| + (x - 4)^2$$

and

$$EI y = -\frac{x^3}{6} + \frac{8}{3} x \bigg| + \frac{(x - 4)^3}{3}$$

**(ii) For portion DC**

Measuring $x$ from $D$, towards right, we get

$$EI \frac{d^2 y}{dx^2} = -M_x = -2 + (1 \times x)$$

$$EI \frac{dy}{dx} = -2x + \frac{x^2}{2} + C_1$$

$$EI y = -x^2 + \frac{x^3}{6} + C_1 x + C_2$$

At $x = 0, \quad (EI \frac{dy}{dx})_{DC} = (EI \frac{dy}{dx})_{DA} = -\frac{34}{3} = C_1$

At $x = 2 \quad \therefore \quad EI y = 0 = -4 + \frac{8}{6} - \frac{34}{3} \times 2 + C_2$
or
\[ C_2 = \frac{76}{3} \]

Hence the slope and deflection equations for portion \( DC \) are
\[ EI \frac{dy}{dx} = -2x + \frac{x^2}{2} - \frac{34}{3} \]  \[ \text{...(III)} \]
and
\[ EIy = -x^3 + \frac{x^3}{6} - \frac{34}{3}x + \frac{76}{3} \]  \[ \text{...(IV)} \]

At \( x = 0 \), \( (EIy)_{DC} = \frac{76}{3} \)

Hence
\[ Y_{dd} = (y_{na}) - (y_{dc}) = \frac{1}{EI} \left( -\frac{52}{3} - \frac{76}{3} \right) = -\frac{128}{3EI} \]

The calculations of \( Y_{xd} \) and \( F_d \) are done in the tabular form below (Table 5.7). The term \( EI \) has been omitted for convenience.

<table>
<thead>
<tr>
<th>Dist. from ( A ) (m)</th>
<th>Eq. No.</th>
<th>( Y_{xd} )</th>
<th>( F_d = \frac{Y_{xd}}{Y_{dd}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( (x = 0) )</td>
<td>II</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1 ( (x = 1) )</td>
<td>II</td>
<td>(-\frac{1}{6} + \frac{8}{3} = +\frac{5}{2})</td>
<td>(-\frac{5 \times 3}{28} = -0.094)</td>
</tr>
<tr>
<td>2 ( (x = 2) )</td>
<td>II</td>
<td>(-\frac{8}{6} + \frac{16}{3} = +4)</td>
<td>(-\frac{7 \times 3}{2 \times 128} = -0.082)</td>
</tr>
<tr>
<td>3 ( (x = 3) )</td>
<td>II</td>
<td>(-\frac{27}{6} + \frac{24}{3} = +\frac{7}{2})</td>
<td>(-\frac{27}{6} + \frac{32}{3} = 0)</td>
</tr>
<tr>
<td>4 ( (x = 4) )</td>
<td>II</td>
<td>(-\frac{64}{6} + \frac{32}{3} = 0)</td>
<td>0</td>
</tr>
<tr>
<td>5 ( (x = 5) )</td>
<td>II</td>
<td>(-\frac{125}{6} + \frac{40}{3} + \frac{1}{3} = -\frac{43}{6})</td>
<td>(+\frac{43}{6} - \frac{3}{128} = +0.168)</td>
</tr>
<tr>
<td>6 (left) ( (x = 6) )</td>
<td>II</td>
<td>(-\frac{216}{6} + \frac{48}{3} + \frac{8}{3} = -\frac{52}{3})</td>
<td>(+\frac{52}{3} - \frac{3}{128} = +0.406)</td>
</tr>
<tr>
<td>6(right) ( (x = 7) )</td>
<td>IV</td>
<td>(+\frac{76}{3})</td>
<td>(-\frac{76}{3} + \frac{3}{128} = -0.594)</td>
</tr>
<tr>
<td>7 ( (x = 1) )</td>
<td>IV</td>
<td>(-\frac{1}{6} + \frac{34}{3} + \frac{76}{3} = +\frac{79}{6})</td>
<td>(-\frac{76}{6} \times \frac{3}{128} = -0.308)</td>
</tr>
<tr>
<td>8 ( (x = 2) )</td>
<td>IV</td>
<td>(-\frac{4}{6} + \frac{8}{6} + \frac{68}{3} + \frac{76}{3} = 0)</td>
<td>0</td>
</tr>
</tbody>
</table>

5.8. FIXED BEAMS
1. I.I.L. for support moment

Let us now take a fixed beam \( AB \), and plot the I.I.L. for support moment \( M_A \) at \( A \). Let the unit load be at section \( X \), distant \( x \) from \( A \) [Fig. 5.17(a)]. As per Müller-Breslau principle, introduce a hinge at \( A \), thus getting a basic determinate structure, which will deflect under
the unit load at $X$, as shown in Fig. 5.17 (b). Let $\phi_{AX}$ be the resulting rotation at end $A$. Now remove the unit load and apply a unit moment at $A$, due to which the beam will deflect as shown in Fig. 5.17 (c). Let $\phi_{AA}$ be the resulting rotation at $A$ and $y'_{XA}$ be the deflection at $x$, due to unit moment applied at $A$. Then

$$M_A \cdot \phi_{AA} = \phi_{AX}$$

But $\phi_{AX} = y'_{XA}$

by reciprocal theorem.

Hence

$$M_A \cdot \phi_{AA} = y'_{XA}$$

or

$$M_A = \frac{y'_{XA}}{\phi_{AA}} \ldots (5.9)$$

Thus, the deflection curve of Fig. 5.17 (c), to some scale, gives the I.L. for $M_A$. If $\phi_{AA}$ is taken as unity, the deflection at any point $X$ gives the I.L. for $M_A$.

Thus, the basic problem is shown in Fig. 5.17 (d), wherein we have to find the value of deflection $y'_{XA}$ at $X$, due to unit moment applied at $A$. We will solve the problem by the conjugate beam method. Let the reactive moment at end $B$ be $M$, due to unit moment applied at end $A$. The component B.M.D. is shown in Fig. 5.17 (e). The conjugate beam $A' B'$ along with the elastic loading (equal to $M/EnI$ diagram) is shown in Fig. 5.17 (f).

For the conjugate beam, $M'_{A'} = 0$, because of the hinge at $A'$.

Hence,

$$M'_{A'} = 0 = -\frac{1}{2} M \cdot L \left[ \frac{2}{3} L \right] + \frac{1}{2} \frac{L}{E I} \times \frac{1}{3} L$$
which gives \( M = \frac{L}{2} \).

For reaction \( R'_{A} \) at \( A' \) take moments at \( B' \) and equate to zero, since end \( B' \) of the conjugate beam is free.

\[
\therefore \quad R'_{A} \times L - \frac{1}{2} \times \frac{L}{EI} \times \frac{2}{3} L + \frac{1}{2} \times \frac{L}{2EI} \times \frac{1}{3} L = 0
\]

which gives

\[
R'_{A} = + \frac{L}{4EI} \quad \text{i.e.} \quad R'_{A} = \frac{L}{4EI} \quad (1)
\]

Now \( M_{x}' \) of the conjugate beam will give \( y_{xA}' \) of the real beam.

\[
\therefore \quad M'_{x} = \frac{L}{4EI} x + \frac{1}{2} \left( \frac{1}{2EI} \frac{x}{L} \right) x \cdot \frac{L}{3} - \frac{1}{6} \left[ \frac{1}{EI} \left( \frac{L-x}{L} \right) + \frac{2}{EI} \right]
\]

or

\[
M'_{x} = \frac{1}{12EI} (3L^2 x - 6x^2 L + 3x^3)
\]

\[
\therefore \quad y_{xA}' = M_{x}' = \frac{1}{12EI} (3L^2 x - 6x^2 L + 3x^3)
\]

Also, \( \phi_{AA} \) of real beam = \( R'_{A} = \frac{L}{4EI} \)

Hence from Eq. 5.9,

\[
M_{A} = \frac{y_{xA}'}{\phi_{AA}} = \frac{1}{12EI} \left[ 3L^2 x - 6x^2 L + 3x^3 \right] \left( \frac{4EI}{L} \right)
\]

or

\[
M_{A} = \frac{\sqrt{x}}{L} \left( L-x \right)^2 \quad \text{...(5.10)}
\]

Let us check this result by taking \( x = a \) and \( L-x=b \) and by taking a point load \( W \) in place of unit load. In that case,

\[
M_{A} = \frac{W \cdot ab^2}{L^3}
\]

which matches with the well known result.

The I.L. for \( M_{A} \) is shown in Fig. 5.17 (g). For finding the maximum value of \( M_{A} \), we have

\[
\frac{dM_{A}}{dx} = 0 = \frac{1}{L} \left[ L^3 + 3x^2 - 4Lx \right]
\]

or

\[
(L-x)(L-3x) = 0
\]

From which, we get \( x = L/3 \).

\[
\therefore \quad M_{A} = \frac{L/3}{L^2} \left[ L - \frac{L}{3} \right]^2 = \frac{4}{27} L
\]

2. I.L. for Support reaction

Let us now plot I.L. for support reaction \( R_{A} \), for a fixed beam shown in Fig. 5.18 (a). Fig. 5.18 (b) shows the basic determinate structure by removing the support reaction \( R_{A} \), but
by keeping fixidity intact at end A through an induced moment. The end A will deflect by an amount $y_{AX}$ due to unit load placed at X.

Now remove the unit load from X, and place it at end A, due to which the beam will deflect by $y_{AA}$ at A and $y_{XA}$ at X, as shown in Fig. 5.18 (c). From the method of consistent deformation,

$$R_A \cdot y_{AA} = y_{AX}$$

But $y_{AX} = y_{XA}$

by reciprocal theorem

$$\therefore R_A \cdot y_{AA} = y_{XA}$$

or

$$R_A = \frac{y_{XA}}{y_{AA}} \quad \ldots(5.11)$$

Thus, the deflection curve of Fig. 5.18(c) gives, to some scale, the I.L. for $R_A$. If $y_{AA}$ is selected as unity, the deflection curve gives the I.L. for $R_A$, in which the ordinate $y_{XA}$ at any point X, due to unit load placed at A, gives the ordinate of I.L. diagram. Thus the basic problem, shown in Fig. 5.18 (d) lies in finding the value of deflection $y_{XA}$ due to unit load and a reactive moment M at end A.

Fig. 5.18 (e) shows the component B.M. diagram for the beam problem of Fig. 5.18 (d), the corresponding conjugate beam $A' B'$ is shown in Fig. 5.18(f), with the elastic loading $(M/EI)$, in which end $A'$ is fixed while end $B'$ is free. Since the slope at A to the real beam, represented by shear at $A'$ of the conjugate beam, is zero, we have $R_A' = 0$ for the conjugate beam.

$$\therefore R_A' = 0 = - \frac{1}{2} \frac{L}{EI} \cdot L + \frac{1}{2} \frac{M}{EI} \cdot L$$

FIG. 5.18.
From which \( M = L \).

Again deflection \( y_{xA} \) of the real beam is given by B.M. \( M'x \) of the conjugate beam.

\[
y_{xA} = M'x
\]

\[
= \frac{1}{2} \left( \frac{L}{EI} \left( \frac{L-x}{L} \right) \right) (L-x) \times \frac{(L-x)^2}{3} = \frac{(L-x)}{6} \left( \frac{x}{EI} + \frac{2L}{EI} \right)
\]

or

\[
y_{xA} = \frac{1}{6EI} (L-x)^3 - \frac{(L-x)^2}{6EI} (x + 2L)
\]

or

\[
y_{xA} = -\frac{(L-x)^2}{6EI} (L + 2x)
\]

...(i)

At \( x = 0 \),

\[
y_{AA} = M' = -\frac{L^3}{6EI}
\]

...(ii)

Now

\[
R_A = \frac{y_{xA}}{y_{AA}}
\]

\[
= \frac{(L-x)^2}{6EI} \frac{(L + 2x)}{L^3}
\]

or

\[
R_A = \frac{(L-x)^2}{L^3} (L + 2x)
\]

...(5.12)

which gives the equation of I.L. for \( R_A \).

At \( x = 0 \), \( R_A = 1 \) (as expected).

**Check.** For a single point load \( W \) acting at \( a \) from \( A \) and \( b \) from \( B \), we have the well known expression

\[
R_A = \frac{Wb^3}{L^3} (3a + b)
\]

Putting \( W = 1 \), \( a = x \) and \( b = L - x \), we get

\[
R_A = \frac{(L-x)^2}{L^3} (3x + L - x) = \frac{(L-x)^2}{L^3} (L + 2x)
\]

which is the same as Eq. 5.12.

Fig. 5.18 (g) shows the I.L. for \( R_A \), which is a third degree curve.

**PROBLEMS**

1. A beam \( ABC \) of uniform section, length \( 2L \), is hinged at the collinear supports at its centre and ends. Derive the equation to the influence lines for bending moment at the central support. Taking \( L = 4 \) m, plot the influence line to scale indicating values at every quarter of each span.
2. A continuous beam $ABC$ is shown in Fig. 5.19. Compute the ordinates of the influence line for the reaction at $C$, at every quarter point of each span.

FIG. 5.19.

3. For the continuous beam shown in Fig. 5.20, draw the influence lines for reaction at $A$, $B$ and $C$. Indicate the values at every at every quarter of each span.

FIG. 5.20

ANSWERS

1. $O(0) = 0$; $O_1 = +0.236$; $O_2 = +0.376$; $O_3 = +0.328$; $O_4 = 0$

   $O(0) = +0.328$; $O_5 = +0.376$; $O_7 = +0.236$; $O_8 = 0$.

2. $O(0) = 0 = O_1 = -0.028$; $O_2 = -0.045$; $O_3 = -0.039$; $O_4 = 0$

   $O(1) = +0.149$; $O_5 = +0.386$; $O_7 = +0.680$; $O_8 = 1$.

3.

<table>
<thead>
<tr>
<th>Ordinate</th>
<th>$O_0$</th>
<th>$O_1$</th>
<th>$O_2$</th>
<th>$O_3$</th>
<th>$O_4$</th>
<th>$O_5$</th>
<th>$O_6$</th>
<th>$O_7$</th>
<th>$O_8$</th>
</tr>
</thead>
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<tr>
<td>$R_A$</td>
<td>+1.00</td>
<td>0.6718</td>
<td>+0.375</td>
<td>+0.1406</td>
<td>0</td>
<td>-0.0273</td>
<td>-0.0312</td>
<td>-0.0196</td>
<td>0</td>
</tr>
<tr>
<td>$R_B$</td>
<td>0.00</td>
<td>0.4844</td>
<td>+0.8750</td>
<td>+1.0781</td>
<td>+1.00</td>
<td>+0.832</td>
<td>+0.5938</td>
<td>+0.3086</td>
<td>0</td>
</tr>
<tr>
<td>$R_C$</td>
<td>0.00</td>
<td>-0.1563</td>
<td>-0.2500</td>
<td>-0.2188</td>
<td>0</td>
<td>+0.1954</td>
<td>+0.4375</td>
<td>+0.7109</td>
<td>1.00</td>
</tr>
</tbody>
</table>
PART-II
STATICALLY INDETERMINATE STRUCTURES

6  STATICALLY INDETERMINATE BEAMS AND FRAMES

7  THE GENERAL METHOD
(METHOD OF CONSISTENT DEFORMATION)

8  THREE MOMENT EQUATION METHOD

9  THE SLOPE DEFLECTION METHOD

10 MOMENT DISTRIBUTION METHOD

11 THE COLUMN ANALOGY METHOD

12 METHOD OF STRAIN ENERGY

13 DEFLECTION OF PERFECT FRAMES

14 REDUNDANT FRAMES

15 CABLES AND SUSPENSION BRIDGES

16 ARCHES

(123)
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members and are overstiff. However, a redundant structure may have both external as well as internal redundancies.

The main difference between the redundant structures and the statically determinate ones resides in the fact that the stress distribution depends for the first ones not only on the loading but also on the relative dimensions of their members and on the properties of materials of which the members are made. Statically indeterminate structures are very sensitive to such factors as the settlement of their supports, temperature variation, lack of fitness of members which give rise to additional stresses, while the same factors would have no influence whatsoever on statically determinate structures. However, statically indeterminate structures are most widely used in various engineering activities.

6.2. TYPES OF SUPPORTS : REACTION COMPONENTS

There are three types of supports which may be encountered in plane structures (i) a roller support, (ii) a hinge support and (iii) a built-in or fixed supports. These are shown in Fig. 6.1 (a), (b) and (c) respectively.

A roller support consists of two rockers—the upper rocker and the lower rocker, with a pin in between permitting the rotation of the upper rocker with respect to the lower one. Both the rockers can move together on rollers along the bearing plate. Such a support supplies a reactive force which acts normal to the surface of rolling and is directed through the centre of the hinged pin. Thus, only one parameter of the reaction, i.e. its magnitude, has to be known in order to determine the reactions completely. Such a support is also known as free end support or simple support.

A hinged support [Fig. 6.1 (b)] differs from the roller support by the fact that the lower rocker is fixed and cannot move. The reaction passes through the centre of the pin but its magnitude and direction is unknown. In other words, this support has two reaction components—the horizontal and the vertical. Schematically, a hinged support is represented by two bars connected by pin, or sometimes simply by a pin.

The built-in-support, shown in Fig. 6.1 (c), has zero degree of freedom. The determination of the reactions developed by this support requires the knowledge of three parameters—the direction and magnitude of a force passing through any chosen point (or its horizontal and vertical components) and the magnitude of the moment about the same point. Thus, a fixed support provides three reaction components.

6.3. EXTERNAL REDUNDANCY

For any structure, supported on external supports, the total reaction components can be easily found. The stability of a structure depends on the number and arrangement of the reaction components and component parts, rather than on the strength of the supports and parts of the structure. In general, three reaction components are necessary for the external stability
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Fig. 6.4 shows a stiff jointed frame. Assuming that the reaction components are known for the purposes of determining internal indeterminancy, and treating column $AB$ as free body, the thrust, shear and moment in $AB$ are known. Consider joint $B$ at which there are total nine unknowns (i.e. thrust, shear and moment each for $BA$, $BC$, and $BE$), out of which three unknowns of $BA$ are known and three conditions of static equilibrium at joint $B$ can be arbitrarily assigned to the three unknowns of $BC$. Thus, there remain three unknowns in the member $BE$ at joint $B$. Once the internal stresses in $BE$ are determined, the total number of unknowns at joint $E$ are six (three for $ED$ and three for $EG$), out of which three equations of statical equilibrium of joint $E$ can be arbitrarily assigned to the unknowns of the member $ED$. Thus there remain three unknowns in the member $EG$ at the joint $E$. Knowing the stress components in $BC$, $ED$ and $GF$, and using the equations of statical equilibrium at joints $C$, $D$ and $F$, the unknown stress components in $CD$ and $FD$ can be easily determined. Thus, on the whole, there are 6 unknowns (3 for $BE$ and 3 for $EG$) to be determined and the frame is internally indeterminate to 6th degree. In general, therefore, the degree of internal indeterminacy $I$ can be represented by the formula:

$$I = 3a \quad \ldots(6.6)$$

where $a$ is the number of areas completely enclosed by members of the frame. In Fig. 6.4, $a = 2$, and hence $I = 3 \times 2 = 6$ as determined above. The above formula is also applicable to continuous beams (Fig. 6.2), where $a = 0$ and hence $I = 0$. i.e., a continuous beam is statically determinate internally since the moment and shear at any point on the beam can be readily determined once the external redundant reactions are determined.

The external indeterminateness is given by Eq. 6.1.

$$E = R - r$$

In the case of stiff jointed frame, $r = 3$, since no hinge or link is provided in between the members. Hence

$$E = R - 3 \quad \ldots(5.7)$$

Hence the total indeterminateness or redundancy is given by:

$$T = E + I = (R - r) + 3a = (R - 3) + 3a \quad \ldots(6.8)$$

Fig. 6.5 shows some stiff-jointed structures.
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Example 7.1. A cantilever of uniform flexural stiffness is propped at the remote end. Find the load on the prop when a load $W$ is applied at the centre of the cantilever.

Solution

Let

$$V_B = \text{Redundant reaction (vertical) at } B.$$  

The basic determinate structure is obtained by replacing the prop at $B$ by an unknown vertical reaction $V_B$ as shown in Fig. 7.1 (b). Let

$$\Delta_B = \text{Deflection of the end } B \text{ of the basic determinate structure due to external loading [Fig. 7.1 (c)].}$$  

$$\delta_{BB} = \text{Deflection of the end } B \text{ of the basic determinate structure due to unit load at } B \text{ (The first suffix denotes the point where the deflection is reckoned and the second suffix denotes the position of the unit point load) [Fig. 7.1(e)].}$$

From conditions of geometry at $B$, we get

$$\Delta_B + V_B \cdot \delta_{BB} = 0 \quad \quad \text{(7.1)}$$

The deflection $\Delta_B$ and $\delta_{BB}$ can be obtained either by the area moment method or the conjugate beam method. Fig. 7.1(c) shows the load $W$ acting on the beam, with $\Delta_B$ as the deflection of the end $B$; the corresponding bending moment diagram is shown in Fig. 7.1(d).

$$\Delta_B = -\frac{1}{EI} \sum \delta_{A} \cdot \delta_{E} = -\frac{1}{EI} \left( -\frac{1}{2} \cdot \frac{W}{L} \cdot \frac{L}{2} \right) \left( \frac{L}{2} + \frac{2}{3} \cdot \frac{L}{2} \right) = \frac{5}{48} \frac{W L^3}{EI}$$

Fig 7.1 (e) and (f) shows the unit load acting at $B$ and the corresponding B.M.D. respectively.

$$\therefore \quad \delta_{BB} = -\frac{1}{EI} \sum \delta_{A} \cdot \delta_{E} = -\frac{1}{EI} \left( +\frac{1}{2} \cdot \frac{L}{L} \cdot \frac{2}{3} \cdot \frac{L}{L} \right) = -\frac{L^3}{3EI}$$

Substituting in Eq. 7.1, we get

$$\frac{5}{48} \frac{W L^3}{EI} - V_B \cdot \frac{L^3}{3EI} = 0 \quad \text{or} \quad V_B = \frac{5}{16} W$$

Alternative Solution: An alternative basic determinate structure can be obtained by treating moment at $A$ as redundant. The basic determinate structure, thus, has end $A$ as simply supported with an unknown moment $M_A$, acting, as shown in Fig. 7.2(b).

Let $\theta_A = \text{slope at end } A \text{ of the basic determinate structure, due to the external loading as shown in Fig. 7.2(c).}$

$$\phi_{AA} = \text{slope at end } A, \text{ of the basic determinate structure, due to unit moment acting at } A \text{ as shown in Fig. 7.2(e).}$$

Then by conditions of geometry at $A,$
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You have either reached a page that is unavailable for viewing or reached your viewing limit for this book.
Hence \[ \delta_{AB} = \int \frac{Mm_{A}dx}{EI} = \int \frac{m_{XB} - m_{XA}}{EI} dx \] \hspace{1cm} ...(2)

Comparing (1) and (2), we get \( \delta_{AB} = \delta_{BA} \).

Similarly, other versions of the reciprocal theorem can also be proved.

7.4. GENERALISED MAXWELL'S THEOREM: BETTI'S RECIPROCAL THEOREM

Generalised Statement. If an elastic system is in equilibrium under one set of forces with their corresponding displacements and if the same system is also in equilibrium under second set of forces acting through the same points with their corresponding displacements, then the product of first group of forces and the corresponding displacements caused by second group is equal to the product of the second group of forces and the corresponding displacements caused by the first group.

i.e. \[ P_{A} \Delta_{A} + P_{B} \Delta_{B} = P_{A}' \Delta_{A} + P_{B}' \Delta_{B} \] \hspace{1cm} ...(7.7)

where \( P \) and \( \Delta \) constitute first group of forces and their corresponding displacements, and \( P' \) and \( \Delta' \) constitute second group of forces and displacements.

That is, the virtual work done by the first set of forces acting through the second set of displacements is equal to the virtual work done by the second set of forces acting through the first set of displacements.

In Betti's theorem, the symbols \( P \) and \( \Delta \) can also denote couples and rotations respectively, as well as forces and linear deflections,

i.e. \[ M_{A} \theta_{A} + M_{B} \theta_{B} = M_{A}' \theta_{A} + M_{B}' \theta_{B} \] \hspace{1cm} ...(7.8)

Thus, according to Betti's law, we have, in general

\[ \Sigma P \Delta' + \Sigma M \theta' = \Sigma P' \Delta + \Sigma M' \theta \] \hspace{1cm} ...(7.9)

Example 7.3. A continuous beam ABC is loaded as shown in Fig. 7.6 (a). Determine all reactions and draw B.M. and S.F. diagrams.

Solution. A basic determinate structure is obtained by replacing the central support by an upward force \( V_{B} \) [Fig. 7.6(b)]. Since there are three unknowns, i.e. \( V_{A}, V_{B} \) and \( V_{C} \), the beam is indeterminate to the first degree and the following condition equation will be used.

\[ \Delta_{B} = V_{B} \delta_{AB} \] (Numerically) \hspace{1cm} ...(1)

Now \[ \Delta_{B} = W \delta_{BD} \]

But from reciprocal deflections, \( \delta_{BD} = \delta_{DB} \)

Hence \[ \Delta_{B} = W \delta_{DB} \]

Substituting in (1), we get the modified condition equation

\[ W \delta_{DB} = V_{B} \delta_{BB} \] \hspace{1cm} ...(2)

When the unit load acts at \( B \) [Fig. 7.6(e)], the B.M. diagram will be a triangle having a maximum ordinate of \( -\frac{L}{2} \) at \( B \). Hence the conjugate beam [Fig. 7.6(f)] loaded with \( \frac{M}{EI} \) diagram will be acted upon by a triangular load acting upwards.

From conjugate beam method, [Fig. 7.6(e)] and \[ \Gamma \text{-fig. 7.6(f)}, \]

\[ EI \delta_{BB} = \text{B.M at } B' \text{ due to loading of Fig. 7.6 (f).} \]
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However, we shall discuss in this chapter, only the first method.

8.2. CLAPEYRON'S THEOREM OF THREE MOMENTS: DERIVATION

Let us take two consecutive spans \( AB \) (\( = L_1 \)) and \( BC \) (\( = L_2 \)) of a continuous beam subjected to a general system of loading, as shown in Fig. 8.2 (a). We will use suffix 1 (i.e. \( L_1, E_1, I_1 \)) for the first span \( AB \) and suffix 2 (i.e. \( L_2, E_2, I_2 \) etc) for the second span \( BC \). Fig. 8.2 (b) shows the deflected position of the two spans, after the supports \( A \), \( B \) and \( C \) have settled to positions \( A', B' \) and \( C' \), by amounts \( \delta_A, \delta_B \) and \( \delta_C \) respectively below the original centre line.

From the deflected position of two spans, we have:

\[
y_B^A = \delta_B - \delta_A = \delta_1 \quad \text{(say)}
\]

where \( y_B^A \) is the deflection of \( B \) with respect to \( A \). Similarly

\[
y_B^C = \delta_B - \delta_C = \delta_2 \quad \text{(say)}
\]

where \( y_B^C \) is the deflection of \( B \) with respect to \( C \).

The \( \mu \) and \( \mu' \) diagrams can be constructed as usual. Fig. 8.1 (c) shows the \( \mu \)-diagrams for the two spans, considering the two spans to be simply supported. Fig. 8.1 (d) shows the \( \mu' \) diagram in which \( M_A, M_B \) and \( M_C \) are the fixing moments at the three points. The values of \( M_A, M_B \) and \( M_C \) are to be determined, using Clapeyron's three moment theorem which we are presently deriving.

For the first span \( AB \), measuring \( x \) positive to the right, we have

\[
- E_1 I_1 \frac{d^2 y}{dx^2} = \mu_x + \mu_x', \quad \text{with usual notations.}
\]

Multiplying both sides by \( x \) and integrating over the range \( x = 0 \) to \( x = L_1 \), we get

\[
- E_1 I_1 \left[ x \frac{dy}{dx} - y \right]_0^{L_1} = \int_0^{L_1} \mu_x x \, dx + \int_0^{L_1} \mu_x' x \, dx.
\]

At \( x = L_1 \),

\[
\frac{dy}{dx} = i_B \quad \text{and} \quad y_B^A = \delta_1
\]

\[
\therefore \quad - E_1 I_1 [L_1 i_B - \delta_1] = A_1 \bar{x}_1 + A_1' \bar{x}_1'
\]

From which

\[
i_B = - \frac{1}{E_1 I_1 L_1} \left[ A_1 \bar{x}_1 + A_1' \bar{x}_1' \right] + \frac{\delta_1}{L_1}
\]

...(1)
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<tr>
<th>S.N.</th>
<th>Type of Loading</th>
<th>$\frac{6A_{EI}}{L}$</th>
<th>$\frac{6A_{ER}}{L}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td><img src="image1" alt="Diagram" /></td>
<td>$\frac{wL^3}{4}$</td>
<td>$\frac{wL^3}{4}$</td>
</tr>
<tr>
<td>2.</td>
<td><img src="image2" alt="Diagram" /></td>
<td>$\frac{8}{60}wL^3$</td>
<td>$\frac{7}{60}wL^3$</td>
</tr>
<tr>
<td>3.</td>
<td><img src="image3" alt="Diagram" /></td>
<td>$\frac{7}{60}wL^3$</td>
<td>$\frac{8}{60}wL^3$</td>
</tr>
<tr>
<td>4.</td>
<td><img src="image4" alt="Diagram" /></td>
<td>$\frac{5}{32}wL^3$</td>
<td>$\frac{5}{32}wL^3$</td>
</tr>
<tr>
<td>5.</td>
<td><img src="image5" alt="Diagram" /></td>
<td>$\frac{W_a}{L}(L^2-a^2)$</td>
<td>$\frac{W_b}{L}(L^2-b^2)$</td>
</tr>
<tr>
<td>6.</td>
<td><img src="image6" alt="Diagram" /></td>
<td>$-\frac{a}{L}(3a^2-L^2)$</td>
<td>$+\frac{a}{L}(3b^2-L^2)$</td>
</tr>
<tr>
<td>7.</td>
<td><img src="image7" alt="Diagram" /></td>
<td>$\frac{w}{4L} \left[ a^2(2L^2-b^2) - a^2(2L^2-a^2) \right]$</td>
<td>$\frac{w}{4L} \left[ b^2(2L^2-a^2) - c^2(2L^2-c^2) \right]$</td>
</tr>
</tbody>
</table>
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For span $BC$

\[ A = \frac{2}{3} \times 5 \times 62.5 = 208.33 \]

With $C$ as origin,
\[ A\bar{x} = 208.33 \times 2.5 = 520.83 \]

With $B$ as origin,
\[ A\bar{x} = 208.33 \times 2.5 = 520.83 \]

For span $CD$

\[ A = \frac{1}{2} \times 5 \times 72 = 180 \]

\[ A\bar{x} = 180 \times \frac{1}{3} (5 + 3) = 480, \text{ with } D \text{ as origin.} \]

Applying three moment theorem equation for spans $AB-BC$

\[ 6M_A + 2M_B(6 + 5) + 5M_C + \frac{6A_1 \bar{x}_1}{6} + \frac{6A_2 \bar{x}_2}{5} = 0 \]

or

\[ 6 \times 0 + 22M_B + 5M_C = - \left[ \frac{6}{6} + 853.36 + \frac{6}{5} \times 520.83 \right] = -1478.4 \]

... (1)

Similarly, applying three moment theorem equation for spans $BC-CD$

\[ 5M_B + 2M_C(5 + 5) + 5M_D + \frac{6A_1 \bar{x}_1}{5} + \frac{6A_2 \bar{x}_2}{5} = 0 \]

or

\[ 5M_B + 20M_C + 5 \times 0 = - \left[ \frac{6}{5} \times 520.83 + \frac{6}{5} \times 480 \right] = -1200.1 \]

... (2)

Solving (1) and (2), we get $M_B = -56.79$ kN-m and $M_C = -45.81$ kN-m

For finding reaction $R_A$, write equation for B.M. at $B$. Thus

\[ R_A \times 6 - 80 \times 4 = M_B = -56.79, \text{ from which } R_A = 43.87 \text{ kN} \]

For finding reaction $R_B$, write equation for B.M. at $C$. Thus

\[ R_A \times 11 - 80 \times 9 + R_B \times 5 - 20 \times 5 \times 2.5 = M_C = -45.81 \]

or

\[ 43.87 \times 11 - 720 + 5R_B - 250 = -45.81, \text{ from which } R_B = 88.32 \text{ kN} \]

For reaction $R_D$, write equation for B.M. at $C$, considering R.H.S.

\[ R_D \times 5 - 60 \times 2 = M_C = -45.81, \text{ from which } R_D = 14.84 \text{ kN} \]

Hence

\[ R_C = (80 + 20 \times 5 + 60) - (43.87 + 88.32 + 14.84) = 92.97 \text{ kN} \]

The B.M. and S.F. diagrams are shown in Fig. 8.9 (c) and (d) respectively.

Example 8.4. Solve example 8.3 if the support $B$ sinks by 5 mm. $I$ for the section is 9300 cm$^4$ and $E = 2.10 \times 10^5$ N/mm$^2$.

Solution:

Applying three moments theorem for spans $AB-BC$, we get

\[ 22M_B + 5M_C + 1478.4 = 6EI \left( \frac{\delta_1}{L_1} + \frac{\delta_2}{L_2} \right) \]

In which $\frac{6A_1 \bar{x}_1}{L_1} + \frac{6A_2 \bar{x}_2}{L_2} = 1478.4$ from example 8.3.

Here $E = 2.1 \times 10^5$ N/mm$^2$; $I = 9300 \times 10^4$ mm$^4$

\[ \therefore EI = 210 \times 9300 \times 10^4 \text{ kN-mm}^2 = 19530 \times 10^6 \text{ kN-mm}^2 = 19530 \text{ kN-mm}^2 \]

\[ \delta_1 = 5 \text{ mm} = \frac{5}{1000} \text{ m} = \frac{1}{200} \text{ m} = \delta_2 \]

\[ \frac{\delta_1}{L_1} + \frac{\delta_2}{L_2} = \frac{1}{200} \text{ m} = \delta_2 \]
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or \[ M_B + 4M_C + M_D = -240 \text{ where } M_D = 0 \]
\[ \therefore \quad M_B + 4M_C = -240 \quad \ldots(3) \]

From (1) and (2), we get \[ M_B + 0.2564M_C = -48.46 \quad \ldots(4) \]

From (3) and (4), we get \[ M_C = -51.16 \text{ kN-m} \]
Hence from (4), we get \[ M_B = -35.34 \text{ kN-m} \]
Lastly from (1), we get \[ M_A = -74.16 \text{ kN-m} \]

For reaction at D, write equation for B.M. at C.
\[ \therefore \quad R_D \times 5 - 60 \times 2 = M_C = -51.16 \text{, from which } R_D = 13.768 \text{ kN} \]

For reaction at C, write equation for B.M. at B
\[ \therefore \quad 13.768 \times 10 + R_C \times 5 - 60 \times 7 - 20 \times 5 \times \frac{5}{2} = M_B = -35.34 \text{, from which } R_C = 99.396 \text{ kN} \]

For reaction at B, write equation for B.M. at A.
\[ \therefore \quad 13.768 \times 16 + 99.396 \times 11 + R_B \times 6 - 60 \times 13 - 100 \times 8.5 - 80 \times 2 = M_A = -74.16 \]

From which we get \[ R_B = 67.033 \text{ kN} \]

For reaction at A, write equation for B.M. at B, considering L.H.S.
\[ R_A \times 6 + M_A - 80 \times 4 = M_B \]

or \[ 6R_A - 74.16 - 320 = -35.34 \text{ from which } R_A = 59.803 \text{ kN} \]

Check

Total load = 80 + 100 + 60 = 240 kN

Total reaction = 59.803 + 67.033 + 99.396 + 13.768 = 240 kN

The B.M. and S.F. diagrams are shown in Fig. 8.14 (b) and (c) respectively.

8.9. ADDITIONAL ILLUSTRATIVE EXAMPLES

Example 8.9. A straight elastic beam of uniform section rests on four similar elastic supports which are placed L apart. The supports are such that they are compressed by d for each unit of load upon them. Show that when a uniformly distributed load of total amount W comes on them, the reaction at central supports are each

\[ \frac{W \left( \frac{11}{6} + \frac{3EI d}{L^3} \right)}{\left( 5 + \frac{12EI d}{L^3} \right)} \quad \text{(Cambridge)} \]

Solution:

Total load = W
\[ \therefore \quad \text{U.D.L.} \quad w = \frac{W}{3L} \]

Let \( R_A, R_B, R_C \) and \( R_D \) be the reactions at \( A, B, C \) and \( D \) respectively. By symmetry, \( R_A = R_D \) and \( R_B = R_C \).
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0 + 2M_A(0 + 8) + M_B(8) + 0 + \frac{6}{8}(853.33) = 0

or
2M_A + M_B + 80 = 0

Writing three moment theorem equation for spans AB - BC, we get
M_A(8) + 2M_B(8 + 5) + M_C(5) + \frac{6}{8}(853.33) + \frac{6}{5}(-173.333) = 0

or
8M_A + 26M_B + 5M_C + 431.998 = 0

Imagine a point C' to the right of C such that CC' = 0.
Hence for spans BC - CC'

M_B(5) + 2M_C(5 + 0) + 0 + \frac{6}{5}(-25.667) + 0 = 0

or
5M_B + 10M_C - 32 = 0

Now from (1) and (2), 22M_B + 5M_C + 111.98 = 0

From (3) and (4), we get $M_A = -6.564$ and $M_C = +6.482 \text{ kN-m}$ (i.e. (5))

Hence from (1), $M_A = -36.718$. The B.M. diagram is shown in Fig. 8.21(c).

Reactions: For reaction at A, write equation for B.M. at B. Thus

$R_A(8) + M_A - 5 \times 8 \times 4 = M_B$

or
$8R_A - 36.718 - 160 = -6.564$ from which $R_A = 23.769 \text{ kN}$ (↑)

For reaction at C, write equation for B.M. at B, considering R.H.S. Thus,

$R_C(5) + M_C - 80 = M_B$

or
$5R_C + 6.482 - 80 = -6.564$, from which $R_C = 13.391 \text{ kN}$ (↑)

For reaction at B, write equation for B.M. at A, considering R.H.S. Thus

$M_C - 80 + R_B \times 8 + R_C \times 13 - 5 \times 8 \times 4 = M_A$

or
$+6.482 - 80 + 8R_B + 13.391 \times 13 - 160 = -36.718$ from which $R_B = 2.840$ (↑)

Check:
Total reaction = 23.769 + 13.391 + 2.840 = 40.0 kN

Total load = $5 \times 8 = 40$ kN

The S.F. diagram is shown in Fig. 8.21 (d).

PROBLEMS

1. Determine the lengths of the overhangs for a continuous beam shown in Fig. 8.22 so that the moments over the supports will be equal.

2. For the continuous beam shown in Fig. 8.23, determine support moments at B and C.
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\[ M_{BA} + M_{AD} - M_{BC} = 0. \]

Hence in the new sign convention that will be used in this method, a support moment acting in the clockwise direction will be taken as positive and that in the anti-clockwise direction as negative. A corresponding change will have to be made while plotting the support moment diagram. For any span of a beam or member with rigid joints, a positive support moment (or end moment) at the right hand end will be plotted above the base line and negative support moment below the base. Similarly, for the left hand end, the negative end moments is plotted above the base and positive end moment is plotted below the base line, as shown in Fig. 9.2.

In addition to the above sign convention for the end moments, the following sign convention for the rotation and settlement is adopted: (1) A clockwise rotation (or slope) will be taken as positive and anti-clockwise rotation as negative. (2) If one end of a beam settles, the settlement (or deflection) will be taken as positive if it rotates the beam as a whole in the clockwise direction, and negative if it rotates the beam as whole in the anti-clockwise direction.

9.2. FUNDAMENTAL EQUATIONS

To derive the fundamental slope deflection equations, consider a beam \( AB \), hinged at ends \( A \) and \( B \), and subjected to external loading as shown in Fig. 9.3. Due to the external loading, the beam will be bent and the ends will rotate. Let us now apply end moments \( M_{FAB} \) and \( M_{FBA} \) at ends \( A \) and \( B \) respectively, of such magnitude that the slopes \( \theta_A \) and \( \theta_B \) at the ends \( A \) and \( B \), due to the external loading are reduced to zero. In other words, the applied moments has action similar to that of fixed end moments and therefore, a suffix \( F \) has been used with these end moments. Such moment, which keeps the slope at the end to zero, will hereafter be called fixed end moment, and can be very easily calculated from the standard fixed-beam-formulæ for a given system of loading on the beam. The suffix \( AB \) to \( M_F \) denotes fixed moment at \( A \) for the beam \( AB \). Similarly, \( M_{FBA} \) denotes the fixing moment at \( B \) for the beam \( BA \).

Let the ends \( A \) and \( B \) rotate through \( \theta_A \) and \( \theta_B \) respectively, and end \( B \) deflects (or settle) downwards by an amount \( \delta \). Thus the ends of the beam are having relative movements, both rotational as well as translational, and the beam takes the form shown by solid line in Fig. 9.3 (b). Let \( m_{AB} \) and \( m_{BA} \) be the additional moments resulting from the displacement of the beam from its initial position when the ends were horizontal and at the same level. The rotation \( \theta_A \) and \( \theta_B \), and the deflection \( \delta \) are all positive. \( AB' \) is the tangent to the curve at the end \( A \), at an angle \( \theta_A \) to the horizontal, while \( B_1A' \) is tangent to the curve at \( B \), at an angle \( \theta_B \) to the horizontal line through \( B \). The additional moments \( m_{AB} \) and \( m_{BA} \), causing rotations and settlement of Fig. 9.3 (b) can be easily calculated by the use of area moment method. Fig. 9.3 (c) shows the component bending moment diagram for these additional moments.

Let, \( Y_B^A \) = deviation of \( B \) with reference to the tangent \( A \). Then from Mohr's theorem,
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and \( M_{CB} = \frac{2EI}{5} (2 \theta_C + \theta_B) + 5.0 \)

\[ = 0.4 EI (2 \theta_C + \theta_B) + 5.0 \quad \ldots (4) \]

(c) **Equilibrium equations**

Since end \( A \) is freely supported,
\[ M_{AB} = 0 \]
\[ \therefore \quad 0.4 EI (2 \theta_A + \theta_B) - 2.4 = 0 \quad \ldots (I) \]

Also end \( C \) is freely supported,
\[ M_{CB} = 0 \]
\[ \therefore \quad 0.4 EI (2 \theta_C + \theta_B) + 5.0 = 0 \quad \ldots (II) \]

For the joint \( B \),
\[ M_{BA} + M_{BC} = 0 \]
\[ \therefore \quad [0.4 EI (2 \theta_B + \theta_A) + 3.6] + [0.4 EI (2 \theta_B + \theta_C) - 5.0] = 0 \]

or
\[ 0.4 EI (4 \theta_B + \theta_A + \theta_C) - 1.4 = 0 \quad \ldots (III) \]

Solving Eqs. I, II and III, we get
\[ EI \theta_A = \frac{22.5}{12} \quad \ldots (i) \]
\[ EI \theta_B = \frac{27}{12} \quad \ldots (ii) \]
\[ EI \theta_C = \frac{88.5}{12} \quad \ldots (iii) \]

(d) **Final moments** : Substituting the values of \( EI \theta_A \) and \( EI \theta_B \) in Eq. (2), we get
\[ M_{BA} = 0.4 \left[ \frac{2 \times 27}{12} + \frac{22.5}{12} \right] + 3.6 = + 6.15 \text{ kN-m} \]

As a check, substituting in Eq. (3)
\[ M_{BC} = 0.4 \left( \frac{2 \times 27}{12} - \frac{88.5}{12} \right) - 5.0 = - 6.15 \text{ kN-m} \]

\[ M_{BA} + M_{BC} = + 6.15 - 6.15 = 0. \]

The bending moment diagram and the deflected shape of the beam are shown in Fig. 9.5 (b) and (c) respectively.

*Note.* The beam is statically indeterminate to single degree only. This problem has also been solved by the moment distribution method (Example 10.2) treating the moment at \( B \) as unknown. However, in the slope-deflection method, the slope or rotations are taken as unknowns, and due to this the problem involves three unknown rotations \( \theta_A, \theta_B \) and \( \theta_C \). Hence the method of slope deflection is not recommended for such a problem.

**Example 9.3.** A continuous beam \( ABCD \) consists of three spans and is loaded as shown in Fig. 9.6 (a). Ends \( A \) and \( D \) are fixed. Determine the bending moments at the supports and plot the bending moment diagram.

**Solution** (a) **fixed end moments**

\[ M_{FAB} = - \frac{2 \times 6^3}{12} = - 6 \text{ kN-m} ; \quad M_{FBA} = + \frac{2 \times 6^3}{12} = + 6 \text{ kN-m} \]

\[ M_{FBC} = - \frac{5 \times 3 \times 2^2}{5^2} = - 2.4 \text{ kN-m} ; \quad M_{FCB} = + \frac{5 \times 2 \times 3^2}{5^2} = + 3.6 \text{ kN-m} \]
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(c) Equilibrium equations

At the joint \( B \), \( M_{BA} + M_{BC} = 0 \)

\[
\frac{4}{L} EI \theta_B + \frac{2}{L} EI \theta_B + \frac{1}{L} EI \theta_C - \frac{wL^2}{3} = 0
\]

or \( \frac{6}{L} EI \theta_B + \frac{1}{L} EI \theta_C - \frac{wL^2}{3} = 0 \)

But by symmetry, \( \theta_C = -\theta_B \)

\[
\frac{5}{L} EI \theta_B = \frac{wL^2}{3}
\]

or \( \frac{EI}{L} \theta_B = \frac{wL^2}{3} \times \frac{1}{5} = \frac{wL^2}{15} \) ... (i)

and \( \frac{EI}{L} \theta_C = -\frac{EI}{L} \theta_B = -\frac{wL^2}{15} \) ... (ii)

(d) Final moments: Substituting these values in Eqs. (1) to (6), we get

\[
M_{AB} = 2 \left( \frac{wL^2}{15} \right) = +\frac{2}{15} wL^2; \quad M_{BA} = 4 \left( \frac{wL^2}{15} \right) = +\frac{4}{15} wL^2
\]

\[
M_{BC} = 2 \left( \frac{wL^2}{15} \right) - \frac{wL^2}{3} = -\frac{4}{15} wL^2; \quad M_{CB} = 2 \left( -\frac{wL^2}{15} \right) + \frac{wL^2}{3} + \frac{wL^2}{3} = +\frac{4}{15} wL^2
\]

\[
M_{CD} = 4 \left( -\frac{wL^2}{15} \right) = -\frac{4}{15} wL^2; \quad M_{DC} = 2 \left( -\frac{wL^2}{15} \right) = -\frac{2}{15} wL^2.
\]

The B.M.D. and deflected shape of the frame are shown in Fig. 9.13.

9.4. PORTAL FRAMES WITH SIDE WAY

In the case of continuous beams etc., the effect of yielding or settlement of support is taken into account by introducing initial fixed end moments. In the case of portal frames, however, the amount of the joint movement or ‘sway’ is not known and forms an additional unknown. The portal frames may sway due to one of the following reasons:

1. Eccentric or unsymmetrical loading on the portal frame.
2. Unsymmetrical outline of portal frame.
3. Different end conditions of the columns of the portal frame.
4. Non-uniform section of the members of the frame.
5. Horizontal loading on the columns of the frame.
6. Settlement of the supports of the frame.
7. A combination of the above.

In such cases, the joint translations become additional unknown quantities. Some additional conditions will, therefore, be required for analysing the frame. The additional conditions of equilibrium are obtained from the consideration of the shear force exerted on the structure by the external loading. The horizontal shear exerted by a member is equal to the algebraic sum of the moments at the ends divided by the length of the member. Thus the horizontal shear resistance of all such members can be found and the algebraic sum of all such forces
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or \[ \frac{4}{3} EI \theta_C - \frac{4}{3} EI \delta + \frac{8}{3} EI \theta_C - \frac{4}{3} EI \delta = 6 EI \theta_C + \frac{9}{2} EI \delta + 3 EI \theta_C + \frac{9}{2} EI \delta \]

or \[30 EI \theta_C = -70 EI \delta \quad \text{or} \quad \theta_C = -\frac{7}{3} \delta\]

Substituting in equation 5, we get \[-\frac{140}{3} \delta + 5 \delta = \frac{24}{EI} \quad \text{or} \quad -\frac{125}{3} \delta = \frac{24}{EI}\]

or \[\delta = -\frac{0.576}{EI} \quad \text{...(i)} \quad \text{and} \quad \theta_C = \frac{7}{3} \times \frac{0.576}{EI} = \frac{1.344}{EI} \quad \text{...(ii)}\]

(d) Final moments

The values of moments may now be found out by substituting the values of \(\theta_C\) and \(\delta\) in equations 1 to 4.

Thus, \[M_{AC} = \frac{2 EI}{3} \left( \frac{1.344}{EI} + \frac{0.576}{EI} \right) = 1.28 \text{ kN-m}\]

\[M_{CA} = \frac{2 EI}{3} \left( \frac{2 \times 1.344}{EI} + \frac{0.576}{EI} \right) = 2.18 \text{ kN-m}\]

\[M_{CB} = \frac{2 EI}{2} \left( \frac{2 \times 1.344}{EI} - 3 \times \frac{0.576}{2 EI} \right) = 1.82 \text{ kN-m}\]

\[M_{BC} = \frac{2 EI}{2} \left( \frac{1.344}{EI} - \frac{3}{2} \times \frac{0.576}{EI} \right) = 0.48 \text{ kN-m}\]

The B.M. diagram and the deflected shape have been shown in Fig. 9.20.

Example 9.11. A portal frame ABCD is hinged at A and fixed at D and has stiff joints at B and C. The loading is as shown in Fig. 9.21. Draw the bending moment diagram and deflected shape of the frame.

Solution

(a) Fixed end moments

\[M_{FBC} = -\frac{6 \times 2}{8} = -1.5 \text{ kN-m}\]

\[M_{FCB} = +1.5 \text{ kN-m}\]

\[M_{PCD} = -\frac{2 \times 4^2}{12} = -\frac{8}{3} \text{ kN-m}\]

\[M_{FDC} = +\frac{8}{3} \text{ kN-m}\]

(b) Slope deflection equations

Let joints B and C move horizontally by \(\delta\). There are four unknowns: \(\theta_A\), \(\theta_B\), \(\theta_C\) and \(\delta\).

Assume all unknowns to be positive,

\[M_{AB} = \frac{2 EI \times 3 I}{2\times 3} (2 \theta_A + \theta_B - \frac{3 \times \delta}{3}) = EI (2 \theta_A + \theta_B - \delta) \quad \text{...(1)}\]

\[M_{BA} = \frac{2 EI \times 3 I}{2\times 3} (2 \theta_B + \theta_A - \frac{3 \delta}{3}) = EI (2 \theta_B + \theta_A - \delta) \quad \text{...(2)}\]

\[M_{BC} = \frac{2 EI}{2} (2 \theta_B + \theta_C) - 1.5 = EI (2 \theta_B + \theta_C) - 1.5 \quad \text{...(3)}\]

\[\text{FIG. 9.21.}\]
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\[ \theta_B = \frac{5}{8\ E\ K} + \frac{3 \times 0.75}{8\ E\ K} + \frac{0.933}{4\ E\ K} = \frac{1.139}{E\ K} \]

and
\[ \theta_D = \frac{0.75}{E\ K} + \frac{0.933}{E\ K} = \frac{0.842}{E\ K} \]

\[(iii) \]

(e) Final moments:
Substituting the values of \( \theta_B \), \( \theta_C \), \( \theta_D \) and \( \delta \) in equations 1 to 5, we get
\[ M_{AB} = 2\ E\ K\left(\frac{1.139}{E\ K} - 1.5\times\frac{0.75}{E\ K}\right) = +0.028 \text{ kN-m} \]
\[ M_{BA} = 2\ E\ K\left(\frac{2 \times 1.139}{E\ K} - 3 \times \frac{0.75}{E\ K}\right) = +2.31 \text{ kN-m} \]
\[ M_{BC} = 2\ E\ K\left(\frac{2 \times 1.139}{E\ K} \times \frac{0.933}{E\ K}\right) = -2.31 \text{ kN-m} \]
\[ M_{CB} = 2\ E\ K\left(\frac{-2 \times 0.933}{E\ K} + \frac{1.139}{E\ K}\right) = +3.55 \text{ kN-m} \]
\[ M_{CD} = 2\ E\ K\left(\frac{-2 \times 0.933}{E\ K} + \frac{0.842}{E\ K}\right) = -3.55 \text{ kN-m} \]

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9.24.

Example 9.13. The frame shown in Fig. 9.25 has fixed ends at A and D. The end A rotates clockwise through \( \frac{0.20}{E\ K} \) radians and the end D slips to the right through \( \frac{0.4}{E\ K} \) units. Find the moments induced in the members of the frame and sketch the deflected shape. Take \( E\ K \) constant.

Solution. Since there is no external loading, there will be no fixed end moments.

When D moves to the right through a known distance \( \Delta \), the joints B and C will move to the right through some unknown distance \( \delta \). The movement \( \delta \) causes rotation of AB and DC and \( \Delta \) causes negative rotation of CD. So, the net rotation of DC with respect to AB is the algebraic sum of rotations caused by \( \delta \) and \( \Delta \). There are thus three unknowns : \( \theta_B \), \( \theta_C \) and \( \delta \).

(a) Slope deflection equations:
\[ M_{AB} = 2\ E\ K\left(2 \times \frac{0.2}{E\ K} + \theta_B - \frac{3 \delta}{3}\right) \]  
\[ M_{BA} = 2\ E\ K\left(2 \theta_B + \frac{0.2}{E\ K} - \frac{3 \delta}{3}\right) \]  
\[ M_{BC} = 2\ E\ K\left(2 \theta_B + \theta_C\right) \]
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Example 23.1. Determine the principal moments of inertia for an unequal angle section 60 × 40 × 6 mm shown in Fig. 23.6.

Solution. Let O be the centroid of the section. Let the X-axis be at a distance $C_X$ from face $PQ$, and Y-axis be at a distance $C_Y$ from face $PR$.

Area $A = A_1 + A_2 = (40 \times 6) + (54 \times 6) = 240 + 324 = 564 \text{ mm}^2$

$C_X = \frac{(40 \times 6 \times 30) + (54 \times 6 \times 33)}{564} = 20.2 \text{ mm}$

$C_Y = \frac{(40 \times 6 \times 20) + (54 \times 6 \times 3)}{564} = 10.2 \text{ mm}$

$I_{PO} = \left(\frac{1}{3} \times 6 \times 60^3\right) + \left(\frac{1}{3} \times 34 \times 6^3\right) = 43.33 \times 10^4 \text{ mm}^4$

$I_{XX} = I_{PO} - A \cdot C_Y^2 = 43.44 \times 10^4 - 564 (20.2)^2 = 20.34 \times 10^4 \text{ mm}^4$

$I_{PR} = \left(\frac{1}{3} \times 56 \times 6^3\right) + \left(\frac{1}{3} \times 6 \times 40^3\right) = 13.19 \times 10^4 \text{ mm}^4$

$I_{YY} = I_{PR} - A \cdot C_X^2 = 13.19 \times 10^4 - 564 (10.2)^2 = 7.33 \times 10^4 \text{ mm}^4$

and

$I_{XY} = I_{XX} - I_{YY} = [240 (20 - 10.2) (3 - 20.2)] + [324 (33 - 20.2) (3 - 10.2)]$

$= -4.05 \times 10^4 - 2.99 \times 10^4 = -7.04 \times 10^4 \text{ mm}^4$

From Fig. 23.4 the positions of principal axes are given by

$$\tan 2\alpha = \frac{2I_{XY}}{I_{XX} - I_{YY}} = \frac{2 \times 7.04 \times 10^4}{(20.34 - 7.33) \times 10^4} = 1.085$$

$$\therefore 2\alpha = 47^\circ 20'$$

or

$$\alpha = 23^\circ 40'$$ (anticlockwise)

$$\frac{I_{XX} + I_{YY}}{2} = \frac{(20.34 + 7.33) \times 10^4}{2} = 13.84 \times 10^4$$

$$\frac{I_{XX} - I_{YY}}{2} = \frac{(20.34 - 7.33) \times 10^4}{2} = 6.5 \times 10^4$$

$$\sqrt{\frac{(I_{XX} - I_{YY})^2}{2} - I_{YY}^2}$$

$$= \sqrt{(6.5 \times 10^4)^2 + (-7.04 \times 10^4)^2}$$

$$= 9.58 \times 10^4$$

Hence from Eqs. 23.5 and 23.6

$I_{UU} = 13.84 \times 10^4 + 9.58 \times 10^4$

$= 23.42 \times 10^4 \text{ mm}^4$

$I_{VV} = 13.84 \times 10^4 - 9.58 \times 10^4$

$= 4.26 \times 10^4 \text{ mm}^2$

Check $I_{XX} + I_{YY} = I_{UU} + I_{VV}$

$(20.34 \times 10^4 + 7.33 \times 10^4)$

$= (23.42 \times 10^4 + 4.26 \times 10^4)$

$= 27.67 \times 10^4 \approx 27.68 \times 10^4$. 

FIG. 23.6.
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are plotted for some key points of the section, a polygon is obtained. Such a polygon is known as Z-polygon, and is very useful in finding out the minimum value of \( Z \) for the section and the corresponding position of the plane of loading.

From Eq. 23.10, the bending stress at any point \( A \) having co-ordinates \( u_A \) and \( v_A \) with reference to the principal axes, is given by

\[
f_b = \frac{M \cos \theta}{I_{uu}} v_A + \frac{M \sin \theta}{I_{vv}} u_A \quad \ldots (1)
\]

(where \( \theta \) is the angle of plane of loading \( OM \) with \( OV \) axis). Hence,

\[
f_b = M \left[ \frac{v_A \cos \theta}{I_{uu}} + \frac{u_A \sin \theta}{I_{vv}} \right] = \frac{M}{Z} \quad \ldots (23.25)
\]

where \( Z \) = section modulus of the section for the point \( A \), given by

\[
\frac{1}{Z} = \frac{\frac{v_A \cos \theta}{I_{uu}} + \frac{u_A \sin \theta}{I_{vv}}}{1} \quad \ldots (23.26)
\]

or

\[
\frac{v_A}{I_{uu}} \cdot Z \cos \theta + u_A \frac{Z \sin \theta}{I_{vv}} = 1 \quad \ldots (23.27)(a)
\]

Putting \( Z \cos \theta = v \) and \( Z \sin \theta = u \), we get

\[
\frac{v_A}{I_{uu}} + u \cdot \frac{u_A}{I_{vv}} = 1 \quad \ldots (23.27)
\]

This is the equation of straight line which gives the variation of \( Z \) with \( \theta \). The straight line \( A_1A_2 \) (Fig. 23.15) is called the Z-line for the point \( A \). The intercepts of this straight line on \( UU \) and \( VV \) axes are \( \frac{I_{vv}}{u_A} \) and \( \frac{I_{uu}}{v_A} \) respectively. Hence in order to draw the Z-line for \( A \), set off \( OA_1 = \frac{I_{vv}}{u_A} \) on \( UU \) axis, and \( OA_2 = \frac{I_{uu}}{v_A} \) on the \( VV \) axis. Join \( A_1A_2 \), which is the Z-line for the point \( A \). The minimum value of \( Z \) is given by the perpendicular \( OA_4 \) inclined at an angle \( \phi \) with the \( OV \) axis.

\[ Z_{min} = OA_4 \]

Maximum bending stress at \( A \) will be \( \frac{M}{OA_4} \) when the plane of bending is inclined at \( \theta \) to the principal axis \( OV \).

It will be useful to plot such Z-lines for some key points of a given section, getting what is known as the Z-polygon. We shall take the case of a rectangular section \( ABCD \) of width \( b \) and depth \( d \), to plot the Z-polygon. (Fig. 23.16). The principal axes \( UU \) and \( VV \) of a rectangular section coincide with the usual \( XX \) and \( YY \)-axes passing through its centroid.

For the Z-line for \( A \), the distance \( OP = \frac{I_{vv}}{u_A} \) and \( OQ = \frac{I_{uu}}{u_A} \).

But

\[
I_{vv} = I_{yy} = \frac{1}{12} bd^3 \quad ; \quad u_A = \frac{b}{2}
\]

\[
I_{uu} = I_{xx} = \frac{1}{12} bd^3 \quad ; \quad v_A = \frac{d}{2}
\]
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About the Book

This book, first published in July 1965, and entering into its Twelfth Edition, is one of the three volumes on Mechanics of Materials and Structures. The book, divided into three sections, contains Twenty Three Chapters. Each topic introduced is thoroughly described, the theory is rigorously developed and a large number of numerical examples are included to illustrate its application. General statements of important principles and methods are almost invariably given by practical illustrations. Many new advanced problems have been added which will be useful for competitive examinations. A large number of problems, along with their answers, are available at the end of each chapter to enable the student to test his reading at different stages of his studies.

Author’s Profile

Dr. B. C. Punmia is an eminent author of 18 books, most of which are followed as text books. Having graduated in 1959 with ‘HONOURS’, he obtained his Master’s degree in 1969 with ‘HONOURS’ and Ph. D. in 1976. Having started his career as Assistant Professor in 1959, he was elevated to the posts of Reader in 1965 and Professor in 1978. He also held the posts of Head of Civil Engg. and Dean of the Faculty of Engineering at M.B.M. Engineering College, Jodhpur. During his teaching career of about 36 years, he guided a large number of research students. Dr. Punmia has authored more than hundred papers, monographs and technical reports published in various Indian and Foreign Journals. He had been consultant to various government, semi-government and private organisations.

Ashok Kumar Jain is an eminent Design and Field Engineer. He is also a prolific writer in the domain of Civil Engineering.

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